

BUNDLE GERBES FOR CHERN-SIMONS AND WESS-ZUMINO-WITTEN THEORIES

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ABSTRACT. We develop the theory of Chern-Simons bundle 2-gerbes and multiplicative bundle gerbes associated to any principal G -bundle with connection and a class in $H^4(BG, \mathbb{Z})$ for a compact semi-simple Lie group G . The Chern-Simons bundle 2-gerbe realises differential geometrically the Cheeger-Simons invariant. We apply these notions to refine the Dijkgraaf-Witten correspondence between three dimensional Chern-Simons functionals and Wess-Zumino-Witten models associated to the group G . We do this by introducing a lifting to the level of bundle gerbes of the natural map from $H^4(BG, \mathbb{Z})$ to $H^3(G, \mathbb{Z})$. The notion of a multiplicative bundle gerbe accounts geometrically for the subtleties in this correspondence for non-simply connected Lie groups. The implications for Wess-Zumino-Witten models are also discussed.

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1. INTRODUCTION

In [42] Quillen introduced the determinant line bundle of Cauchy-Riemann operators on a Hermitian vector bundle coupled to unitary connections over a Riemann surface. This work influenced the development of many lines of investigation including the study of Wess-Zumino-Witten actions on Riemann surfaces. Note that Quillen's determinant line bundle also plays an essential role in the construction of the universal bundle gerbe in [15], see also [8].

The relevance of Chern-Simons gauge theory has been noted by many authors, starting with Ramadas-Singer-Weitsman [43] and recently Dupont-Johansen [20], who used gauge covariance of the Chern-Simons functional to give a geometric

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construction of Quillen line bundles. The curvatures of these line bundles in an analytical set-up were studied extensively by Bismut-Freed [3] and in dimension two, went back to the Atiyah-Bott work on the Yang-Mills equations over Riemann surfaces. [2].

A new element was introduced into this picture by Freed [25] and [24] (a related line of thinking was started by some of the present authors [14]) through the introduction of higher algebraic structures (2-categories) to study Chern-Simons functionals on 3-manifolds with boundary and corners. For closed 3-manifolds one needs to study the behaviour of the Chern-Simons action under gluing formulae (that is topological quantum field theories) generalising the corresponding picture for Wess-Zumino-Witten. Heuristically, there is a Chern-Simons line bundle as in [43], such that for a 3-manifold with boundary, the Chern-Simons action is a section of the Chern-Simons line bundle associated to the boundary Riemann surface. For a codimension two submanifold, a closed circle, the Chern-Simons action takes values in a $U(1)$ -gerbe or an abelian group-like 2-category.

Gerbes first began to enter the picture with J-L Brylinski [6] and Breen [5]. The latter developed the notion of a 2-gerbe as a sheaf of bicategories extending Giraud's [29] definition of a gerbe as a sheaf of groupoids. J-L Brylinski used Giraud's gerbes to study the central extensions of loop groups, string structures and the relation to Deligne cohomology. With McLaughlin, Brylinski developed a 2-gerbe over a manifold M to realise degree 4 integral cohomology on M in [10] and introduced an expression of the 2-gerbe holonomy as a Cheeger-Simons differential character on any manifold with a triangulation. This is the starting point for Gomi [31],[32] who developed a local theory of the Chern-Simons functional along the lines of Freed's suggestion. A different approach to some of these matters using simplicial manifolds has been found by Dupont and Kamber [21].

Our contribution is to develop a global differential geometric realization of Chern-Simons functionals using a Chern-Simons bundle 2-gerbe and to apply this to the question raised by Dijkgraaf and Witten about the relation between Chern-Simons and Wess-Zumino-Witten models. Our approach provides a unifying perspective on all of this previous work in a fashion that can be directly related to the physics literature on Chern-Simons field theory (thought of as a path integral defined in terms of the Chern-Simons functional).

In [23] it is shown that three dimensional Chern-Simons gauge theories with gauge group G can be classified by the integer cohomology group $H^4(BG, \mathbb{Z})$, and conformally invariant sigma models in two dimension with target space a compact Lie group (Wess-Zumino-Witten models) can be classified by $H^3(G, \mathbb{Z})$. It is also established that the correspondence between three dimensional Chern-Simons gauge theories and Wess-Zumino-Witten models is related to the transgression map

$$\tau : H^4(BG, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z}),$$

which explains the subtleties in this correspondence for compact, semi-simple non-simply connected Lie groups ([36]).

In the present work we introduce Chern-Simons bundle 2-gerbes and the notion of multiplicative bundle gerbes, and apply them to explore the geometry of the Dijkgraaf-Witten correspondence. To this end, we will assume throughout that G is a compact semi-simple Lie group.

The role of Deligne cohomology as an ingredient in topological field theories goes back to [27] and we add a new feature in section 2 by using Deligne cohomology

valued characteristic classes for principal G -bundles with connection. Briefly speaking, a degree p Deligne characteristic class for principal G -bundles with connection is an assignment to any principal G -bundle with connection over M of a class in the degree p Deligne cohomology group $H^p(M, \mathcal{D}^p)$ satisfying a certain functorial property. Deligne cohomology valued characteristic classes refine the characteristic classes for principal G -bundles.

We will define three dimensional Chern-Simons gauge theories $CS(G)$ as degree 3 Deligne cohomology valued characteristic classes for principal G -bundles with connection, but will later show that there is a global differential geometric structure, the Chern-Simons bundle 2-gerbe, associated to each Chern-Simons gauge theory. We will interpret a Wess-Zumino-Witten model as arising from the curving of a bundle gerbe associated to a degree 2 Deligne cohomology class on the Lie group G as in [12] and [28]. We then use a certain canonical G bundle defined on $S^1 \times G$ to construct a transgression map between classical Chern-Simons gauge theories $CS(G)$ and classical Wess-Zumino-Witten models $WZW(G)$ in section 3, which is a lift of the transgression map $H^4(BG, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z})$. The resulting correspondence

$$\Psi : CS(G) \longrightarrow WZW(G)$$

refines the Dijkgraaf-Witten correspondence between three dimensional Chern-Simons gauge theories and Wess-Zumino-Witten models associated to a compact Lie group G . On Deligne cohomology groups, our correspondence Ψ induces a transgression map

$$H^3(BG, \mathcal{D}^3) \longrightarrow H^2(G, \mathcal{D}^2),$$

and refines the natural transgression map $\tau : H^4(BG, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z})$ (Cf. Proposition 3.4). See [9] for a related transgression of Deligne cohomology in a different set-up.

For any integral cohomology class in $H^3(G, \mathbb{Z})$, there is a unique stable equivalence class of bundle gerbe ([37, 38]) whose Dixmier-Douady class is the given degree 3 integral cohomology class. Geometrically $H^4(BG, \mathbb{Z})$ can be regarded as stable equivalence classes of bundle 2-gerbes over BG , whose induced bundle gerbe over G has a certain multiplicative structure.

To study the geometry of the correspondence Ψ , we revisit the bundle 2-gerbe theory developed in [44] and [34] in Section 4. Note that transformations between stable isomorphisms provide 2-morphisms making the category \mathbf{BGrb}_M of bundle gerbes over M and stable isomorphisms between bundle gerbes into a *bi-category* (Cf. [44]).

For a smooth surjective submersion $\pi : X \rightarrow M$, consider the face operators $\pi_i : X^{[n]} \rightarrow X^{[n-1]}$ on the simplicial manifold $X_\bullet = \{X_n = X^{[n+1]}\}$. Then a bundle 2-gerbe on M consists of the data of a smooth surjective submersion $\pi : X \rightarrow M$ together with

- (1) An object $(\mathcal{Q}, Y, X^{[2]})$ in $\mathbf{BGrb}_{X^{[2]}}$.
- (2) A stable isomorphism $m : \pi_1^* \mathcal{Q} \otimes \pi_3^* \mathcal{Q} \rightarrow \pi_2^* \mathcal{Q}$ in $\mathbf{BGrb}_{X^{[3]}}$ defining the bundle 2-gerbe product which is associative up to a 2-morphism ϕ in $\mathbf{BGrb}_{X^{[4]}}$.
- (3) The 2-morphism ϕ satisfies a natural coherency condition in $\mathbf{BGrb}_{X^{[5]}}$.

We then develop a multiplicative bundle gerbe theory over G in section 5 as a simplicial bundle gerbe on the simplicial manifold associated to BG . We say a bundle gerbe \mathcal{G} over G is *transgressive* if the Deligne class of \mathcal{G} , written $d(\mathcal{G})$ is in the image of the correspondence map $\Psi : CS(G) \rightarrow WZW(G) = H^2(G, \mathcal{G}^2)$.

The main results of this paper are the following two theorems (Theorem 5.8 and Theorem 5.9)

- (1) *The Dixmier-Douady class of a bundle gerbe \mathcal{G} over G lies in the image of the transgression map $\tau : H^4(BG, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z})$ if and only if \mathcal{G} is multiplicative.*
- (2) *Let \mathcal{G} be a bundle gerbe over G with connection and curving, whose Deligne class $d(\mathcal{G})$ is in $H^2(G, \mathcal{D}^2)$. Then \mathcal{G} is transgressive if and only if \mathcal{G} is multiplicative.*

Let ϕ be an element in $H^4(BG, \mathbb{Z})$. The corresponding G -invariant polynomial on the Lie algebra under the universal Chern-Weil homomorphism is denoted by Φ . For any connection \mathbb{A} on the universal bundle $EG \rightarrow BG$ with the curvature form $F_{\mathbb{A}}$,

$$(\phi, \Phi(\frac{i}{2\pi} F_{\mathbb{A}})) \in H^4(BG, \mathbb{Z}) \times_{H^4(BG, \mathbb{R})} \Omega_{cl,0}^4(BG)$$

(where $\Omega_{cl,0}^4(BG)$ is the space of closed 4-forms on BG with periods in \mathbb{Z}), defines a unique degree 3 Deligne class in $H^3(BG, \mathcal{D}^3)$. Here we fix a smooth infinite dimensional model of $EG \rightarrow BG$ by embedding G into $U(N)$ and letting EG be the Stiefel manifold of N orthonormal vectors in a separable complex Hilbert space.

We will show that $H^3(BG, \mathcal{D}^3)$ classifies the stable equivalence classes of bundle 2-gerbes with curving on BG , (we already know that the second Deligne cohomology classifies the stable equivalence classes of bundle gerbes with curving). These are the universal Chern-Simons bundle 2-gerbes \mathcal{Q}_ϕ in section 6 (cf. Proposition 6.4) giving a geometric realisation of the degree 3 Deligne class determined by $(\phi, \Phi(\frac{i}{2\pi} F_{\mathbb{A}}))$.

We show that for any principal G -bundle P with connection A over M , the associated Chern-Simons bundle 2-gerbe $\mathcal{Q}_\phi(P, A)$ over M is obtained by the pull-back of the universal Chern-Simons bundle 2-gerbe \mathcal{Q}_ϕ via a classifying map. The bundle 2-gerbe curvature of $\mathcal{Q}_\phi(P, A)$ is given by $\Phi(\frac{i}{2\pi} F_A)$, and the bundle 2-gerbe curving is given by the Chern-Simons form associated to (P, A) and ϕ .

Under the canonical isomorphism between Deligne cohomology and Cheeger-Simons cohomology, there is a canonical holonomy map for any degree p Deligne class from the group of smooth p -cocycles to $U(1)$. This holonomy is known as the Cheeger-Simons differential character associated to the Deligne class.

The bundle 2-gerbe holonomy for this Chern-Simons bundle 2-gerbe $\mathcal{Q}_\phi(P, A)$ over M as given by the Cheeger-Simons differential character is used in the integrand for the path integral for the Chern-Simons quantum field theory. In the $SU(N)$ Chern-Simons theory, Φ is chosen to be the second Chern polynomial. For a smooth map $\sigma : Y \rightarrow M$, under a fixed trivialisation of $\sigma^*(P, A)$ over Y , the corresponding holonomy of σ is given by $e^{2\pi i CS(\sigma, A)}$, where $CS(\sigma, A)$ can be written as the following well-known Chern-Simons form:

$$\frac{k}{8\pi^2} \int_Y Tr \sigma^*(A \wedge dA + \frac{1}{3} A \wedge A \wedge A),$$

Here $k \in \mathbb{Z}$ is the level determined by $\phi \in H^4(BSU(N), \mathbb{Z}) \cong \mathbb{Z}$.

We will establish in Theorem 6.7 that the Chern-Simons bundle 2-gerbe $\mathcal{Q}_\phi(P, A)$ over M is equivalent in Deligne cohomology to the Cheeger-Simons invariant associated to the principal G -bundle P with a connection A and a class $\phi \in H^4(BG, \mathbb{Z})$.

In the concluding section we show that the Wess-Zumino-Witten models in the image of this correspondence Ψ satisfy a quite interesting multiplicative property that is associated with the group multiplication on G . This multiplicative property is a feature of the holonomy of every multiplicative bundle gerbe. It implies that for the transgressive Wess-Zumino-Witten models, the so-called B field satisfies a certain integrality condition. Using our multiplicative bundle gerbe theory, we can give a very satisfying explanation why, for non-simply connected groups, multiplicative bundle gerbes only exist for Dixmier Douady classes that are certain particular multiples of the generator in $H^3(G, \mathbb{Z})$ (we call this the ‘level’). For a non-simply connected Lie group, there exists a subtlety in the construction of positive energy representations of its loop group, see [41] [45], where the level is defined in terms of its Lie algebra.

While this paper was in preparation, Aschieri and Jurčo in [1] proposed a similar construction of Chern-Simons 2-gerbes in terms of Deligne classes developed in [34] to study M5-brane anomalies and E_8 gauge theory. Their discussions have some overlaps with our local descriptions of bundle 2-gerbes and holonomy of 2-gerbes.

2. DELIGNE CHARACTERISTIC CLASSES FOR PRINCIPAL G -BUNDLES

In this Section, we first review briefly Deligne and Cheeger-Simons cohomology, and then define a Deligne cohomology valued characteristic class for any principal G -bundle with connection over a smooth manifold M with G a compact semi-simple Lie group.

Let $H^p(M, \mathcal{D}^p)$ be the p -th Deligne cohomology group, which is the hypercohomology group of the complex of sheaves on M :

$$\underline{U(1)} \xrightarrow{d\log} \Omega_M^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_M^p$$

where $\underline{U(1)}$ is the sheaf of smooth $U(1)$ -valued functions on M , Ω_M^p is the sheaf of imaginary-valued differential p -forms on M . Take any degree p Deligne class

$$\xi = [g, \omega^1, \dots, \omega^p],$$

then with respect to a good cover of M , $\{g_{i_0 i_1 \dots i_p}\}$ represents a $U(1)$ -valued Čech p -cocycle on M , and hence defines an element in

$$H^p(M, \underline{U(1)}) \cong H^{p+1}(M, \mathbb{Z}).$$

The corresponding element in $H^{p+1}(M, \mathbb{Z})$ is denoted by $c(\xi)$, and referred to as the *characteristic class* of ξ . Moreover, $d\omega^p$ is a globally defined closed $p+1$ form on M with periods in $2\pi i \mathbb{Z}$ called the *curvature* of ξ and denoted by $curv(\xi)$. Without causing any confusion, we often identify $curv(\xi)$ with $curv(\xi)/2\pi i$ whose periods are in \mathbb{Z} .

We have indexed the Deligne cohomology group so that a degree p Deligne class has holonomy (to be discussed later in this section) over p dimensional sub-manifolds and a characteristic class in $H^{p+1}(M, \mathbb{Z})$. For example, $H^1(M, \mathcal{D}^1)$ is the space of equivalence classes of line bundles with connection, whose holonomy is defined for any smooth path and whose characteristic class in $H^2(M, \mathbb{Z})$ is given by the first Chern class of the underlying line bundle. Next in the hierarchy, $H^2(M, \mathcal{D}^2)$ is the space of stable isomorphism classes of bundle gerbes with connection and curving, whose holonomy is defined for any 2-dimensional closed sub-manifold and

whose characteristic class in $H^3(M, \mathbb{Z})$ is given by the Dixmier-Douady class of the underlying bundle gerbe.

The Deligne cohomology group $H^p(M, \mathcal{D}^p)$ is part of the following two exact sequences (Cf. [7])

$$(2.1) \quad 0 \rightarrow \Omega_{cl,0}^p(M) \rightarrow \Omega^p(M) \rightarrow H^p(M, \mathcal{D}^p) \xrightarrow{c} H^{p+1}(M, \mathbb{Z}) \rightarrow 0$$

where $\Omega_{cl,0}^p(M)$ is the subspace of closed p -forms on M with periods in \mathbb{Z} , in the space of p -forms $\Omega^p(M)$, and c is the characteristic class map; and

$$(2.2) \quad 0 \rightarrow H^p(M, \mathbb{R}/\mathbb{Z}) \rightarrow H^p(M, \mathcal{D}^p) \xrightarrow{curv} \Omega_{cl,0}^{p+1}(M) \rightarrow 0$$

where the map $curv$ is the curvature map on $H^p(M, \mathcal{D}^p)$.

We remark that for a Deligne class $\xi \in H^p(M, \mathcal{D}^p)$, $curv(\xi)$ the curvature of ξ defines a cohomology class in $H^{p+1}(M, \mathbb{R})$ which agrees with the image of $c(\xi)$ under the map $H^{p+1}(M, \mathbb{Z}) \rightarrow H^{p+1}(M, \mathbb{R})$ sending an integral class to a real class.

Recall that the Cheeger-Simons group of differential characters of degree p in [16], $\check{H}^p(M, U(1))$, is defined to be the space of pairs, (χ, ω) consisting of a homomorphism

$$\chi : Z_p(M, \mathbb{Z}) \rightarrow U(1)$$

where $Z_p(M, \mathbb{Z})$ is the group of smooth p -cycles, and an imaginary-valued closed $(p+1)$ -form ω on M with periods in $2\pi i\mathbb{Z}$ such that for any smooth $(p+1)$ chain σ

$$\chi(\partial\sigma) = \exp\left(\int_\sigma \omega\right).$$

The Cheeger-Simons group $\check{H}^p(M, U(1))$ enjoys the same exact sequences (2.1) and (2.2) as the Deligne cohomology group $H^p(M, \mathcal{D}^p)$. In fact, the holonomy and the curvature of a Deligne class define a canonical isomorphism

$$(2.3) \quad (hol, curv) : H^p(M, \mathcal{D}^p) \longrightarrow \check{H}^p(M, U(1)).$$

Here the holonomy of a Deligne class ξ is defined as follows. For a smooth p -cycle given by a triangulation of a smooth map $X \rightarrow M$, pull back ξ to X to obtain a Deligne class on X . Lift this class to an element α in $\Omega^p(X)$ from the exact sequence (2.1) as $H^{p+1}(X, \mathbb{Z}) = 0$ and then

$$hol(\xi) = \exp\left(\int_X \alpha\right)$$

is independent of the choice of α , this again follows from (2.1). For a general smooth p -cycle $\sigma = \sum_k n_k \sigma_k$, we choose a local representative $(g, \omega^1, \dots, \omega^p)$ of ξ under a good cover $\{U_i\}$ of M such that for each smooth p -simplex $f_k : \sigma_k \rightarrow M$, with possible subdivisions, $f_k(\sigma_k)$ is contained in some open set U_{i_k} for which f_k has a smooth extension. We can define (see [27] for $p = 2$, and [13, 31] for $p = 2, 3$)

$$\begin{aligned} \mathcal{A}(\sigma_k) &= \exp\left\{\int_{\sigma_k} f_k^* \omega_{i_k}^p + \sum_{\tau_{(1)}} \int_{\tau_{(1)} \subset \sigma_k} f_k^* \omega_{i_{\tau_{(1)}} i_k}^{p-1} + \dots\right\} \\ &\cdot \prod_{\tau_{(p)} \in \tau_{(p-1)} \subset \dots \subset \tau_{(1)} \subset \sigma_k} g_{i_k i_{\tau_{(1)}} \dots i_{\tau_{(p)}}} \end{aligned}$$

where $\{\tau_{(j)}\}$ is the set of codimension j faces of σ_k , with $\tau_{(j)} \subset U_{i_{\tau_{(j)}}}$ and induced orientation from σ_k . It is routine to show that

$$hol(\xi) = \prod_k \mathcal{A}(\sigma_k)^{n_k}$$

is independent of local representative of ξ and subdivisions because $\sum_k n_k \sigma_k$ is a cycle.

With the understanding of (2.3), we often identify a Deligne class with the corresponding Cheeger-Simons differential character. In [33], Hopkins and Singer develop a cochain model for the Cheeger-Simons cohomology where certain integrations are well-defined, see also [22]. In this paper, we need to define an integral on the Deligne cohomology group:

$$(2.4) \quad \int_{S^1} : \quad H^3(S^1 \times G, \mathcal{D}^3) \rightarrow H^2(G, \mathcal{D}^2).$$

While this makes sense under the map $(hol, curv)$ and Hopkins-Singer's integration for the cochain model for the Cheeger-Simons differential characters we will instead apply the exact sequences (2.1) and (2.2) to uniquely define the integration map (2.4) via the following two commutative diagrams:

$$(2.5) \quad \begin{array}{ccccccc} H^3(S^1 \times G, \mathbb{R}/\mathbb{Z}) & \rightarrow & H^3(S^1 \times G, \mathcal{D}^3) & \xrightarrow{\text{curv}} & \Omega_{cl,0}^4(S^1 \times G) \\ \downarrow \int_{S^1} & & \downarrow \int_{S^1} & & \downarrow \int_{S^1} \\ H^2(G, \mathbb{R}/\mathbb{Z}) & \rightarrow & H^2(G, \mathcal{D}^2) & \xrightarrow{\text{curv}} & \Omega_{cl,0}^3(G), \end{array}$$

where the integration map (2.4) is well-defined modulo the image of $H^2(G, \mathbb{R}/\mathbb{Z}) \rightarrow H^2(G, \mathcal{D}^2)$; and

$$(2.6) \quad \begin{array}{ccccccc} \frac{\Omega^3(S^1 \times G)}{\Omega_{cl,0}^3(S^1 \times G)} & \rightarrow & H^3(S^1 \times G, \mathcal{D}^3) & \xrightarrow{c} & H^4(S^1 \times G, \mathbb{Z}) \\ \downarrow \int_{S^1} & & \downarrow \int_{S^1} & & \downarrow \int_{S^1} \\ \frac{\Omega^2(G)}{\Omega_{cl,0}^2(G)} & \rightarrow & H^2(G, \mathcal{D}^2) & \xrightarrow{c} & H^3(G, \mathbb{Z}), \end{array}$$

where the integration map (2.4) is well-defined modulo the image of $\Omega^2(G) \rightarrow H^2(G, \mathcal{D}^2)$.

To see how these two exact sequences may be used to uniquely specify the integration map we let $r: H^p(M, \mathbb{Z}) \rightarrow H^p(M, \mathbb{R})$ be the map that sends an integral class to a real class. Cheeger and Simons [16] define

$$(2.7) \quad R^p(M, \mathbb{Z}) = \{(\omega, u) \in \Omega_{cl,0}^p(M) \oplus H^p(M, \mathbb{Z}) \mid r(u) = [\omega]\}$$

where $[\omega]$ is the real cohomology class of the differential form ω . This enables them to combine the map of a Deligne class to its characteristic class and the map to its curvature into one map from the Deligne cohomology group into R^{p+1} . There is a short exact sequence:

$$(2.8) \quad 0 \rightarrow \frac{H^p(M, \mathbb{R})}{r(H^p(M, \mathbb{Z}))} \rightarrow H^p(M, \mathcal{D}^p) \xrightarrow{(\text{curv}, c)} R^{p+1}(M, \mathbb{Z}) \rightarrow 0.$$

Then the following induced diagram is commutative:

$$\begin{array}{ccccccc} \frac{H^3(S^1 \times G, \mathbb{R})}{r(H^3(S^1 \times G, \mathbb{Z}))} & \rightarrow & H^3(S^1 \times G, \mathcal{D}^3) & \xrightarrow{(\text{curv}, c)} & R^4(S^1 \times G, \mathbb{Z}) \\ \downarrow \int_{S^1} & & \downarrow \int_{S^1} & & \downarrow \int_{S^1} \\ 0 & \rightarrow & H^2(G, \mathcal{D}^2) & \xrightarrow{(\text{curv}, c)} & R^3(G, \mathbb{Z}) & \rightarrow 0. \end{array}$$

This commutative diagram and the fact that $H^2(G, \mathbb{R}) = 0$ for any compact semi-simple Lie group G is the reason the integration map (2.4) is well defined.

With these preparations we may now define Deligne characteristic classes for principal G -bundles with connection. Recall that a characteristic class c for principal G -bundles is an assignment of a class $c(P) \in H^*(M, \mathbb{Z})$ to every isomorphism class of principal G -bundles $P \rightarrow M$. Of course we could do \mathbb{Q} , \mathbb{R} etc instead of \mathbb{Z} . This assignment is required to be ‘functorial’ in the following sense: if $f: N \rightarrow M$ is a smooth map we require that $c(f^*(P)) = f^*(c(P))$, where f^*P is the pull-back principal G -bundle over N .

It is a standard fact that characteristic classes are in bijective correspondence with elements of $H^*(BG, \mathbb{Z})$. The proof is: given a characteristic class c , we have, of course, $c(EG) \in H^*(BG, \mathbb{Z})$ and conversely if $\xi \in H^*(BG, \mathbb{Z})$ is given, then defining $c_\xi(P) = f^*(\xi)$ for any classifying map $f: M \rightarrow BG$ gives rise to a characteristic class for the isomorphism class of principal G -bundles defined by the classifying map f . This uses the fact that any two classifying maps f and g are homotopy equivalent so that $f^* = g^*$ and hence $f^*(\xi) = g^*(\xi)$. This definition motivates our definition of Deligne characteristic classes for principal G -bundles with connection.

Note that the characteristic classes only depend on the underlying topological principal G -bundle, in order to define a Deligne cohomology valued characteristic class, we will restrict ourselves to differentiable principal G -bundles.

Definition 2.1. A Deligne characteristic class d (of degree p) for principal G -bundles with connection is an assignment to any principal G -bundle P with connection A over M of a class $d(P, A) \in H^p(M, \mathcal{D}^p)$ which is functorial in the sense that if $f: N \rightarrow M$ then

$$d(f^*(P), f^*(A)) = f^*(d(P, A)),$$

where $f^*(P)$ is the pull-back principal G -bundle with the pull-back connection $f^*(A)$.

Note that if we add two Deligne characteristic classes using the group structure in $H^p(M, \mathcal{D}^p)$ the result is another Deligne characteristic class. Denote by $\mathcal{D}_p(G)$ the group of all Deligne characteristic classes of degree p for principal G -bundles.

If $d \in \mathcal{D}_p(G)$ is a Deligne characteristic class for principal G -bundles and $P \rightarrow M$ is a principal G -bundle, we can choose a connection A on P , then $d(P, A) \in H^p(M, \mathcal{D}^p)$. Composing with the characteristic class map for Deligne cohomology

$$c: H^p(M, \mathcal{D}^p) \rightarrow H^{p+1}(M, \mathbb{Z}),$$

we get

$$(2.9) \quad c(d(P, A)) = c \circ d(P, A) \in H^{p+1}(M, \mathbb{Z}).$$

Lemma 2.2. *The above map (2.9) defines a homomorphism $\mathcal{D}_p(G) \rightarrow H^{p+1}(BG, \mathbb{Z})$.*

Proof. If we can show that (2.9) is independent of the choice of connections then we have defined a characteristic class for P , which corresponds to an element in $H^{p+1}(BG, \mathbb{Z})$. Here we approximate BG by finite dimensional smooth models (see [40]), and use the fact that $H^*(BG, \mathbb{Z})$ is the inductive limit of the cohomology of these models. To see that $c \circ d(P, A)$ doesn’t depend of the choice of connections, let A_0 and A_1 be two connections on P and consider the connection \mathbb{A} on $\hat{P} = P \times \mathbb{R} \rightarrow M \times \mathbb{R}$ given by

$$\mathbb{A} = (1-t)A_0 + tA_1.$$

Let $\iota_t : M \rightarrow M \times \mathbb{R}$ be the inclusion map $\iota_t(m) = (m, t)$. It is well known that the induced maps on cohomology $\iota_t^* : H^p(M \times \mathbb{R}, \mathbb{Z}) \rightarrow H^p(M, \mathbb{Z})$ are all equal and isomorphisms. Moreover $\iota_0^*(\hat{P}, \mathbb{A}) = (P, A_0)$ and $\iota_1^*(\hat{P}, \mathbb{A}) = (P, A_1)$ which imply

$$\begin{aligned} c \circ d(P, A_0) &= c \circ d(\iota_0^*(\hat{P}, \mathbb{A})) = \iota_0^* c \circ d(\hat{P}, \mathbb{A}) \\ &= \iota_1^* c \circ d(\hat{P}, \mathbb{A}) = c \circ d(\iota_1^*(\hat{P}, \mathbb{A})) \\ &= c \circ d(P, A_1) \in H^{p+1}(M, \mathbb{Z}). \end{aligned}$$

Hence we have defined a homomorphism

$$\mathcal{D}_p(G) \rightarrow H^{p+1}(BG, \mathbb{Z}).$$

□

Define $I^k(G)$ to be the ring of invariant polynomials on the Lie algebra of G . Then we have the Chern-Weil homomorphism:

$$cw : I^k(G) \rightarrow H^{2k}(BG, \mathbb{R}).$$

If G is compact then this is an isomorphism. Define

$$A^{2k}(G, \mathbb{Z}) = \{(\Phi, \phi) \in I^k(G) \times H^{2k}(BG, \mathbb{Z}) \mid cw(\Phi) = r(\phi)\}.$$

In [16] Cheeger and Simons show that each $(\Phi, \phi) \in A^{2k}(G, \mathbb{Z})$ defines a differential character valued characteristic class of degree $2k-1$, whose value on a principal G -bundle P over M with connection A is denoted by

$$S_{\Phi, \phi}(P, A) \in \check{H}^{2k-1}(M, U(1))$$

(Cf. Remark 6.1). Let $c_{\Phi, \phi}(P, A)$ be the element in $H^{2k-1}(M, \mathcal{D}^{2k-1})$ such that under the natural isomorphism

$$\begin{aligned} \check{H}^{2k-1}(M, U(1)) &\rightarrow H^{2k-1}(M, \mathcal{D}^{2k-1}) \\ S_{\Phi, \phi}(P, A) &\mapsto c_{\Phi, \phi}(P, A). \end{aligned}$$

For the category of principal G -bundles with connection whose morphisms are connection preserving bundle morphisms, then $c_{\Phi, \phi}(P, A)$ is a Deligne characteristic class, which is a functorial lifting of

$$\left(\Phi \left(\frac{i}{2\pi} F_A \right), \phi(P) \right) \in \Omega_{cl, 0}^{2k}(M) \times H^{2k}(M, \mathbb{Z})$$

where F_A is the curvature 2-form of the connection A and $\phi(P)$ is the corresponding characteristic class of P . This defines a map: $A^{2k}(G, \mathbb{Z}) \rightarrow \mathcal{D}_{2k-1}(G)$.

In particular, if G is compact and $\phi \in H^{2k}(BG, \mathbb{Z})$ then

$$\Phi = cw^{-1}(r(\phi)) \in I^k(G)$$

satisfies $cw(\Phi) = r(\phi)$, which means $(\Phi, \phi) \in A^{2k}(G, \mathbb{Z})$. As Φ is determined by ϕ in this case we write $c_\phi \equiv c_{\Phi, \phi}$. So we have a composed map

$$H^{2k}(BG, \mathbb{Z}) \rightarrow A^{2k}(G, \mathbb{Z}) \rightarrow \mathcal{D}_{2k-1}(G)$$

which sends ϕ to c_ϕ .

Proposition 2.3. *For a compact Lie group G , each element ϕ in $H^{2k}(BG, \mathbb{Z})$ defines a degree $2k-1$ Deligne characteristic class c_ϕ in $\mathcal{D}_{2k-1}(G)$ such that $c_\phi \mapsto \phi$ under the homomorphism in Lemma 2.2.*

Proof. From the above discussion and the definition of the Deligne characteristic class, we obtain that, given a principal G -bundle P with a connection A over M , $c_\phi(P, A) = c_{\Phi, \phi}(P, A)$ is a functorial lifting of

$$\left(\Phi\left(\frac{i}{2\pi}F_A\right), \phi(P)\right) \in \Omega_{cl,0}^{2k}(M) \times H^{2k}(M, \mathbb{Z}).$$

This implies that the corresponding characteristic class of P is $\phi(P)$. From the bijective correspondence between degree $2k$ characteristic classes of principal G -bundles and elements of $H^{2k}(BG, \mathbb{Z})$, we know that $c_\phi \mapsto \phi$ under the homomorphism in Lemma 2.2. \square

Fixing a smooth infinite dimensional model of $EG \rightarrow BG$ by embedding the compact semi-simple Lie group G into $U(N)$ and letting EG be the Stiefel manifold of N orthonormal vectors in a separable complex Hilbert space, we know that the Deligne cohomology group $H^{2k-1}(BG, \mathcal{D}^{2k-1})$ is well-defined.

Let \mathbb{A} be a universal connection on EG , $\phi \in H^{2k}(BG, \mathbb{Z})$ defines a degree $2k-1$ Deligne characteristic class

$$c_{\Phi, \phi}(EG, \mathbb{A}) \in H^{2k-1}(BG, \mathcal{D}^{2k-1}),$$

where $\Phi \in I^k(G)$ satisfies $cw(\Phi) = r(\phi)$. Then the following commutative diagram

$$\begin{array}{ccc} H^{2k-1}(BG, \mathcal{D}^{2k-1}) & \longrightarrow & H^{2k}(BG, \mathbb{Z}) \\ \downarrow & & \downarrow \\ \Omega_{cl,0}^{2k}(BG) & \longrightarrow & H^{2k}(BG, \mathbb{R}) \end{array}$$

shows that the map $\phi \mapsto c_\phi(EG, \mathbb{A})$ refines the Chern-Weil homomorphism.

3. FROM CHERN-SIMONS TO WESS-ZUMINO-WITTEN

Let G be a compact, connected, semi-simple Lie group. In [23] Dijkgraaf and Witten discuss a correspondence map between three dimensional Chern-Simons gauge theories and Wess-Zumino-Witten models associated to the compact Lie group G from the topological actions viewpoint, which naturally involves the transgression map

$$\tau : H^4(BG, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z}).$$

(To be precise, τ is actually the inverse of the transgression in Borel's study of topology of Lie groups and characteristic classes but this is of no real importance.)

We recall the definition of τ . We take a class $\phi \in H^4(BG, \mathbb{Z})$ and pull its representative ϕ back to $\pi^*(\phi)$, a four-cocycle on EG . As EG is contractible we have that $\phi = d\tau_\phi$ for a three-cocycle τ_ϕ on EG . Restricting τ_ϕ to a fibre which we identify with G it is easy to show that the result is a closed cocycle defining an element of $H^3(G, \mathbb{Z})$ and that moreover this cohomology class is independent of all choices made.

It is shown in [23] that three dimensional Chern-Simons gauge theories with gauge group G can be classified by the integer cohomology group $H^4(BG, \mathbb{Z})$, and conformally invariant sigma models in two dimension with target space a compact Lie group (Wess-Zumino-Witten models) can be classified by $H^3(G, \mathbb{Z})$. Recall the

commutative diagram

$$\begin{array}{ccc} H^3(BG, \mathcal{D}^3) & \longrightarrow & H^4(BG, \mathbb{Z}) \\ \downarrow & & \downarrow \\ \Omega_{cl,0}^4(BG) & \longrightarrow & H^4(BG, \mathbb{R}). \end{array}$$

To classify the exponentiated Chern-Simons action in three dimensional Chern-Simons gauge theories, we propose the following mathematical definitions of a three dimensional Chern-Simons gauge theory and a Wess-Zumino-Witten model.

Definition 3.1. We make the following definitions:

- (1) A three dimensional *Chern-Simons gauge theory* with gauge group G is defined to be a Deligne characteristic class of degree 3 for a principal G -bundle with connection. We denote the group of all three dimensional Chern-Simons gauge theories with gauge group G by $CS(G)$.
- (2) A *Wess-Zumino-Witten model* on G is defined to be a Deligne class on G of degree 2 and we denote the group of all such by $WZW(G)$.

In brief, $CS(G) = \mathcal{D}_3(G)$ and $WZW(G) = H^2(G, \mathcal{D}^2)$.

With these preliminaries taken care of we can now explain our refined geometric definition of the Dijkgraaf-Witten map and discuss its image. Firstly, we give a more geometric definition of the transgression map τ , which has the advantage that it can be lifted to define a correspondence map from $CS(G)$ to $WZW(G)$. We do this by constructing a canonical G -bundle \mathcal{P} with connection \mathbb{A} on the manifold $S^1 \times G$. It follows that if $d \in CS(G)$ then $d(\mathcal{P}, \mathbb{A}) \in H^3(S^1 \times G, \mathcal{D}^3)$. We can integrate along S^1 with the Deligne characteristic class in $H^3(S^1 \times G, \mathcal{D}^3)$ to get the required map

$$\int_{S^1} d(\mathcal{P}, \mathbb{A}) \in H^2(G, \mathcal{D}^2) = WZW(G).$$

We want to show that for any G there is a natural G bundle over $S^1 \times G$ with connection. To do this it is convenient to work with *pre-G-bundles*.

Definition 3.2. A *pre-G-bundle* is a pair (Y, \hat{g}) where $\pi: Y \rightarrow M$ is a surjective submersion and $\hat{g}: Y^{[2]} \rightarrow G$ such that

$$\hat{g}(y_1, y_3) = \hat{g}(y_1, y_2)\hat{g}(y_2, y_3)$$

for any y_1, y_2, y_3 all in the same fibre of $\pi: Y \rightarrow M$. Here we denote by $Y^{[p]}$ the p -fold fibre product of $\pi: Y \rightarrow M$.

If $P \rightarrow M$ is a principal G bundle, there is a canonical map $\hat{g}: P^{[2]} \rightarrow G$ defined by

$$p_1\hat{g}(p_1, p_2) = p_2.$$

Then (P, \hat{g}) is a pre- G -bundle. Conversely if (Y, \hat{g}) is a pre- G -bundle over M , we can construct a principal G -bundle $P \rightarrow M$ as follows. Take $Y \times G$ and define

$$(y_1, h_1) \sim (y_2, h_2) \quad \text{if } \pi(y_1) = \pi(y_2) \text{ and } h_1\hat{g}(y_1, y_2) = h_2.$$

The space of equivalence classes $P = Y \times G / \sim$ is a principal G -bundle over M with right G -action on equivalence classes $[y, g]$ given by $[y, h] \cdot g = [y, hg]$

Two pre-\$G\$-bundles \$(Y, \hat{g}_1)\$ and \$(X, \hat{g}_2)\$ give rise to isomorphic principal \$G\$ bundles if and only if there is an \$\hat{h}: Y \times_{\pi} X \rightarrow G\$ such that

$$\hat{h}(y_1, x_1)\hat{g}_1(y_1, y_2) = \hat{g}_2(x_1, x_2)\hat{h}(y_2, x_2)$$

for any collection of points \$y_1, y_2 \in Y\$, \$x_1, x_2 \in X\$ mapping to the same point in \$M\$. A pre-\$G\$-bundle \$(Y, \hat{g})\$ is *trivial* if there is an \$\hat{h}: Y \rightarrow G\$ such that

$$\hat{g}(y_1, y_2) = \hat{h}(y_1)^{-1}\hat{h}(y_2)$$

for every \$(y_1, y_2) \in Y^{[2]}\$.

Given a pre-\$G\$-bundle \$(Y, \hat{g})\$ over \$M\$, we denote by \$\hat{g}^{-1}d\hat{g}\$ the pull-back by \$\hat{g}: Y^{[2]} \rightarrow G\$ of the Maurer-Cartan form. Then \$\hat{g}^{-1}d\hat{g}\$ is a \$\mathfrak{g}\$ (the Lie algebra of \$G\$)-valued. Let \$A\$ be a \$\mathfrak{g}\$-valued one-form on \$Y\$. We say that \$A\$ is a connection for the pre-\$G\$-bundle \$(Y, \hat{g})\$ if

$$\pi_1^*(A) = ad(\hat{g})\pi_2^*(A) - \hat{g}^{-1}d\hat{g}$$

where \$\pi_1, \pi_2: Y^{[2]} \rightarrow Y\$ are the projections and we denote the adjoint action of \$G\$ on its Lie algebra by \$ad(\hat{g})\$. It is easy to check that there is a one-to-one correspondence between connections on a pre-\$G\$-bundle and connections on the associated principal \$G\$ bundle.

We wish to define a \$G\$ bundle on \$S^1 \times G\$ with connection. From the previous discussion it suffices to define a pre-\$G\$-bundle with connection. Let \$\mathcal{A}\$ be all smooth maps \$h\$ from \$\mathbb{R}\$ to \$G\$ with \$h^{-1}dh\$ periodic and \$h(0) = 1\$. Define \$\pi: \mathcal{A} \rightarrow G\$ by \$\pi(h) = h(1)\$. Notice that if \$\pi(g) = \pi(h)\$ then \$g = h\gamma\$ where \$\gamma\$ is a smooth map from \$[0, 1]\$ to \$G\$ with \$\gamma^{-1}d\gamma\$ periodic and \$\gamma(1) = 1 = \gamma(0)\$. Such a \$\gamma\$ is actually a smooth based loop in the based loop group \$\Omega G\$. We can identify \$\mathcal{A}\$ with the space of \$G\$-connections on the circle \$S^1 = \mathbb{R}/\mathbb{Z}\$. Then \$\mathcal{A}\$ is contractible and \$\pi: \mathcal{A} \rightarrow G\$ is the holonomy map. Hence, \$\pi: \mathcal{A} \rightarrow G\$ is a universal \$\Omega G\$-bundle, and \$G\$ is a classifying space \$B\Omega G\$ of \$\Omega G\$. Let \$Y = \mathcal{A} \times S^1 \rightarrow G \times S^1\$. Define \$\hat{g}: Y^{[2]} \rightarrow G\$ by

$$\hat{g}(h_1, h_2, \theta) = h_1(\theta)^{-1}h_2(\theta).$$

Then the pair \$(Y, \hat{g})\$ is a pre-\$G\$-bundle over \$G \times S^1\$. Let \$\hat{Y} = \mathbb{R} \times \mathcal{A}\$ and the projection \$\hat{Y} \rightarrow Y\$ induced by \$\mathbb{R} \rightarrow S^1\$. Define \$\hat{h}: \hat{Y} \rightarrow G\$ by \$\hat{h}(t, h) = h(t)\$. \$\hat{h}^{-1}d\hat{h}\$ being periodic descends to a \$\mathfrak{g}\$-valued one-form on \$Y\$. It is straightforward to check that this defines a connection \$A\$ for the pre-\$G\$-bundle \$(Y, \hat{g})\$ over \$G \times S^1\$.

The principal \$G\$-bundle over \$G \times S^1\$ corresponding to the pre-\$G\$-bundle \$Y = \mathcal{A} \times S^1 \rightarrow G \times S^1\$ can be obtained as follows (Cf. [11] and [39]). Denote by

$$\mathcal{P} = \frac{\mathcal{A} \times S^1 \times G}{\Omega G},$$

the quotient space of \$\Omega G\$-action on \$\mathcal{A} \times S^1 \times G\$, where the \$\Omega G\$-action is given by, for \$\gamma \in \Omega G\$ and \$(h, \theta, g) \in \mathcal{A} \times S^1 \times G\$,

$$(3.1) \quad \gamma \cdot (h, \theta, g) = (h\gamma, \theta, \gamma(\theta)^{-1}g).$$

Notice that \$\mathcal{P}\$ admits a natural free \$G\$-action from the right multiplication on \$G\$-factor. The connection \$A\$ on the pre-\$G\$-bundle \$(Y, \hat{g})\$ defines a natural connection \$\mathbb{A}\$ on \$\mathcal{P}\$.

Definition 3.3. The canonical principal G -bundle over $G \times S^1$ is given by \mathcal{P} with connection \mathbb{A} . The correspondence map from three dimensional Chern-Simons gauge theories $CS(G)$ to Wess-Zumino-Witten models $WZW(G)$ is defined to be

$$CS(G) = \mathcal{D}_3(G) \ni d \mapsto \int_{S^1} d(\mathcal{P}, \mathbb{A}) \in H^2(G, \mathcal{D}^2) = WZW(G).$$

Denote this map by $\Psi : CS(G) \longrightarrow WZW(G)$.

The next proposition shows that the map Ψ descends to the natural transgression map from $H^4(BG, \mathbb{Z})$ to $H^3(G, \mathbb{Z})$ and hence Ψ refines the Dijkgraaf-Witten correspondence.

Proposition 3.4. *The correspondence map from $CS(G)$ to $WZW(G)$ induces the natural transgression map $\tau : H^4(BG, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z})$.*

Proof. We first give another construction of τ . Let $EG \rightarrow BG$ be the universal G -bundle, then (as is well known) the ΩG bundle

$$\tilde{\pi} : \Omega EG \rightarrow \Omega BG$$

formed by applying the based loop functor to $EG \rightarrow BG$ gives another model of the universal ΩG -bundle. In particular we have a homotopy equivalence $\Omega BG \xrightarrow{\sim} G$ which lifts to an ΩG -equivariant homotopy equivalence $\Omega EG \xrightarrow{\sim} \mathcal{A}$. This leads to the isomorphism:

$$H^3(\Omega BG, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z}).$$

On the other hand, the natural evaluation map:

$$ev : \Omega BG \times S^1 \longrightarrow BG$$

defines a pull-back map

$$ev^* : H^4(BG, \mathbb{Z}) \rightarrow H^4(\Omega BG \times S^1, \mathbb{Z}),$$

from which the integration along S^1 gives rise to another construction of the transgression map:

$$(3.2) \quad \int \circ ev^* : H^4(BG, \mathbb{Z}) \rightarrow H^3(\Omega BG, \mathbb{Z}) \cong H^3(G, \mathbb{Z}).$$

From the homotopy equivalence between the two universal ΩG -bundles: $\pi : \mathcal{A} \rightarrow G$ and $\tilde{\pi} : \Omega EG \rightarrow \Omega BG$, we get a homotopy equivalence of two principal G -bundles:

$$(3.3) \quad \mathcal{P} = \frac{\mathcal{A} \times S^1 \times G}{\Omega G} \sim \frac{\Omega EG \times S^1 \times G}{\Omega G}.$$

Here the ΩG action on $\Omega EG \times S^1 \times G$ is given by the similar action on $\mathcal{A} \times S^1 \times G$ as in (3.1). $\frac{\Omega EG \times S^1 \times G}{\Omega G}$ is a principal G -bundle over $\Omega BG \times S^1$.

Now we show that the pull-back of the universal G -bundle: $EG \rightarrow BG$ via the evaluation map ev , which is

$$(3.4) \quad ev^*(EG) = (\Omega BG \times S^1) \times_{BG} EG$$

is isomorphic to $\frac{\Omega EG \times S^1 \times G}{\Omega G}$ as principal G -bundles. The isomorphism map

$$(3.5) \quad \frac{\Omega EG \times S^1 \times G}{\Omega G} \rightarrow (\Omega BG \times S^1) \times_{BG} EG$$

is given by

$$[(\tilde{\gamma}, \theta, g)] \mapsto [(\tilde{\pi}(\tilde{\gamma}), \theta, \tilde{\gamma}(\theta) \cdot g)].$$

Here $(\tilde{\gamma}, \theta, g) \in \Omega EG \times S^1 \times G$, $\tilde{\pi}$ is the map $\Omega EG \rightarrow \Omega BG$, $\tilde{\gamma}(\theta)$ is the image of the evaluation map on $\Omega EG \times S^1$ and the action of g on $\tilde{\gamma}(\theta)$ is induced from the right G -action on the universal G : $EG \rightarrow BG$.

It is easy to check that (3.5) is a well-defined G -bundle isomorphism by direct calculation:

$$\begin{aligned} [(\tilde{\gamma} \cdot \gamma, \theta, \gamma(\theta)^{-1}g)] &\mapsto [(\tilde{\pi}(\tilde{\gamma} \cdot \gamma), \theta, (\tilde{\gamma}\gamma)(\theta) \cdot (\gamma(\theta)^{-1}g))] \\ &= [(\tilde{\pi}(\tilde{\gamma}), \theta, \tilde{\gamma}(\theta) \cdot g)], \end{aligned}$$

and for $g' \in G$

$$[(\tilde{\gamma}, \theta, gg')] \mapsto [(\tilde{\pi}(\tilde{\gamma}), \theta, \tilde{\gamma}(\theta) \cdot gg')].$$

Then from (3.3), (3.4) and (3.5), we obtain a homotopy equivalence of two principal G -bundles:

$$(3.6) \quad \begin{array}{ccc} \mathcal{P} & & ev^*(EG) \\ \downarrow & \sim & \downarrow \\ S^1 \times G & & S^1 \times \Omega BG. \end{array}$$

Hence, with the definition of the integration map (2.4) given by (2.5) and (2.6), we see that the transgression map $\tau = \int oev^* : H^4(BG, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z})$ agrees with the map induced by our correspondence $\Psi : CS(G) \rightarrow WZW(G)$. \square

For a compact Lie group, Proposition 2.3 tells us that there exists a one-to-one map

$$CS(G) \rightarrow H^3(BG, \mathcal{D}^3)$$

and the exact sequence (2.1) implies the exact sequence

$$0 \rightarrow \Omega^2(G)/\Omega_{cl,0}^2(G) \rightarrow WZW(G) \rightarrow H^3(G, \mathbb{Z}) \rightarrow 0.$$

As in general, the map $\tau : H^4(BG, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z})$ is not surjective, the map

$$\Psi : CS(G) \rightarrow WZW(G)$$

is not surjective either, for a general compact semi-simple Lie group. We will see that the Wess-Zumino-Witten models from the image of Ψ exhibit some special properties by exploiting bundle gerbe theory.

We give a summary of how various bundle gerbes enter. First it is now well understood how, given a WZW model, we can define an associated bundle gerbe over the group G , as $WZW(G) = H^2(G, \mathcal{D}^2)$ is the space of stable isomorphism classes of bundle gerbes with connection and curving [12, 38]. We will see in the Section 4 that $H^4(BG, \mathbb{Z})$ is the space of stable equivalence classes of bundle 2-gerbes on BG . Thus an element in $H^4(BG, \mathbb{Z})$ defines a class of bundle 2-gerbes on M associated to a principal G -bundle over M using the pullback construction of the classifying map. The corresponding transgressed element in $H^3(G, \mathbb{Z})$ defines a bundle gerbe over G . In fact, as $H^3(BG, \mathbb{R}) = 0$, we know that the third Deligne cohomology group $H^3(BG, \mathcal{D}^3)$ is determined by the following commutative diagram:

$$\begin{array}{ccc} H^3(BG, \mathcal{D}^3) & \xrightarrow{c} & H^4(BG, \mathbb{Z}) \\ \downarrow curv & & \downarrow \\ \Omega_{cl,0}^4(BG) & \longrightarrow & H^4(BG, \mathbb{R}). \end{array}$$

In the last Section we showed that given an element $\phi \in H^4(BG, \mathbb{Z})$, we can take a G -invariant polynomial $\Phi \in I^2(G)$ corresponding to ϕ , then for a connection \mathbb{A} on the universal bundle EG over BG ,

$$(\phi, \Phi(\frac{i}{2\pi} F_{\mathbb{A}})) \in H^4(BG, \mathbb{Z}) \times_{H^4(BG, \mathbb{R})} \Omega_{cl,0}^4(BG)$$

represents a degree 3 Deligne class

$$c_{\phi}(EG, \mathbb{A}) \in H^3(BG, \mathcal{D}^3).$$

In a following Section we will define a universal Chern-Simons bundle 2-gerbe determined by $c_{\phi}(EG, \mathbb{A})$. We will then pull it back to define, for a principal G -bundle P with connection A over M , a Chern-Simons bundle 2-gerbe over M , with 2-curving given by the Chern-Simons form associated to (P, A) .

Remark 3.5. Brylinski [7] defines a generalisation of $H^3(G, \mathbb{Z})$, called the differentiable cohomology, denoted $H_{diff}^3(G, U(1))$ for which there is an isomorphism $H_{diff}^3(G, U(1)) \cong H^4(BG, \mathbb{Z})$. In low dimensions these differentiable cohomology classes have the following interpretations [7]:

$$\begin{aligned} H_{diff}^1(G, U(1)) &\cong \text{smooth homomorphisms } G \rightarrow U(1) \\ H_{diff}^2(G, U(1)) &\cong \text{isomorphism classes of central extensions of } G \text{ by } U(1) \\ H_{diff}^3(G, U(1)) &\cong \text{equivalence classes of multiplicative } U(1)\text{-gerbes on } G. \end{aligned}$$

His multiplicative $U(1)$ -gerbes motivated us to define multiplicative bundle gerbes.

4. BUNDLE 2-GERBES

Bundle 2-gerbe theory on M is developed in [44]. A bundle 2-gerbe with connection and curving, defines a degree 3 Deligne class in $H^3(M, \mathcal{D}^3)$. In [34] it is shown that the group of stable equivalence classes of bundle 2-gerbes with connection and curving is isomorphic to $H^3(M, \mathcal{D}^3)$. We review the results in [44] and [34], then we define multiplicative bundle gerbes on compact Lie group.

We begin with the definition of a simplicial bundle gerbe as in [44] on a simplicial manifold $X_{\bullet} = \{X_n\}_{n \geq 0}$ with face operators $d_i : X_{n+1} \rightarrow X_n$ ($i = 0, 1, \dots, n+1$). We remark that the simplicial manifolds we use in this paper are not required to have degeneracy operators (see [19]).

Definition 4.1. (Cf. [44]) A simplicial bundle gerbe on a simplicial manifold X_{\bullet} consists of the following data.

- (1) A bundle gerbe \mathcal{G} over X_1 .
- (2) A bundle gerbe stable isomorphism $m : d_0^* \mathcal{G} \otimes d_2^* \mathcal{G} \rightarrow d_1^* \mathcal{G}$ over X_2 , where $d_i^* \mathcal{G}$ is the pull-back bundle gerbe over X_2 .
- (3) The bundle gerbe stable isomorphism m is associative up to a natural transformation, called an associator,

$$\phi : d_2^* m \circ (d_0^* m \otimes Id) \rightarrow d_1^* m \circ (Id \otimes d_3^* m)$$

between the induced stable isomorphisms of bundle gerbes over X_3 . The line bundle \mathcal{L}_{ϕ} over X_3 induced by ϕ admits a trivialisation section s such that $\delta(s)$ agrees with the canonical trivialisation of

$$d_0^* \mathcal{L}_{\phi} \otimes d_1^* \mathcal{L}_{\phi}^* \otimes d_2^* \mathcal{L}_{\phi} \otimes d_3^* \mathcal{L}_{\phi}^* \otimes d_4^* \mathcal{L}_{\phi}.$$

If in addition the bundle gerbe \mathcal{G} is equipped with a connection and a curving, and m is a stable isomorphism of bundle gerbes with connection and curving, we call it a simplicial bundle gerbe with connection and curving on X_\bullet .

Remark 4.2. For those unfamiliar with 2-gerbes we offer the following amplification.

- (1) A bundle gerbe stable isomorphism m in Definition 4.1 is a fixed trivialisation of the bundle gerbe

$$\delta(\mathcal{G}) = d_0^* \mathcal{G} \otimes d_1^* \mathcal{G}^* \otimes d_2^* \mathcal{G}$$

over X_2 , where $d_1^* \mathcal{G}^*$ is the dual bundle gerbe of $d_1^* \mathcal{G}$. (See [37][38] for various operations on bundle gerbes and the definition of a bundle gerbe stable isomorphism.)

- (2) With the understanding of m in Definition 4.1 as a fixed trivialisation of the bundle gerbe $d_0^* \mathcal{G} \otimes d_1^* \mathcal{G}^* \otimes d_2^* \mathcal{G}$ over X_2 , we can see that $d_2^* m \circ (d_0^* m \otimes Id)$ and $d_1^* m \circ (Id \otimes d_3^* m)$ represent two trivialisations of the bundle gerbe over X_3 . This induces a line bundle over X_3 (Cf. [37]), called the ϕ -induced or associator line bundle. A simplicial bundle gerbe \mathcal{G} requires that this associator line bundle is trivial and the trivialisation section s satisfies the natural cocycle condition.
- (3) A simplicial bundle gerbe with connection and curving has in its definition, a restrictive condition, as it requires that the bundle gerbe stable isomorphism m preserves connections and curvings. This implies,

$$(4.1) \quad d_0^*(\text{curv}(\mathcal{G})) - d_1^*(\text{curv}(\mathcal{G})) + d_2^*(\text{curv}(\mathcal{G})) = 0.$$

For the simplicial bundle gerbe constructed in this paper, the underlying bundle gerbe is often equipped with a connection and curving, but we shall not require that the bundle gerbe stable isomorphism m preserves the connection and curving. In section 7, instead of (4.1), we have

$$d_0^*(\text{curv}(\mathcal{G})) - d_1^*(\text{curv}(\mathcal{G})) + d_2^*(\text{curv}(\mathcal{G})) = dB,$$

for some 2-form B on X_2 , which is not necessarily obtained from a 2-form on X_1 by the δ -map $d_0^* - d_1^* + d_2^*$.

For a smooth submersion $\pi : X \rightarrow M$, there is a natural associated simplicial manifold $X_\bullet = \{X_n\}$ (which one might well think of as the ‘nerve’ associated to $\pi : X \rightarrow M$) with X_n given by

$$X_n = X^{[n+1]} = X \times_M X \times_M \cdots \times_M X$$

the $(n+1)$ -fold fiber product of π , and face operators $d_i = \pi_{i+1} : X_n \rightarrow X_{n-1}$ ($i = 0, 1, \dots, n$) are given by the natural projections from $X^{[n+1]}$ to $X^{[n]}$ by omitting the entry in i position for π_i . For an exception, we denote by $EG^{[\bullet]}$ the associated simplicial manifold $\{EG^{[n]}\}$ for the universal bundle $\pi : EG \rightarrow BG$.

Now we recall the definition of bundle 2-gerbe on a smooth manifold M from [44] and [34].

Definition 4.3. A bundle 2-gerbe on M consists of a quadruple of smooth manifolds $(\mathcal{Q}, Y; X, M)$ where $\pi : X \rightarrow M$ is a smooth, surjective submersion, and $(\mathcal{Q}, Y; X^{[2]})$ is a simplicial bundle gerbe on the simplicial manifold $X_\bullet = \{X_n = X^{[n+1]}\}$ associated to $\pi : X \rightarrow M$.

It is sometimes convenient to describe bundle 2-gerbes using the language of 2-categories (see for example [35]). One first observes that transformations between stable isomorphisms provide 2-morphisms making the category \mathbf{BGrb}_M of bundle gerbes over M and stable isomorphisms between bundle gerbes into a weak 2-category or *bi-category* (Cf. [44]). Note that the space of 2-morphisms between two stable isomorphisms is one-to-one corresponding to the space of line bundles over M .

Consider the face operators $\pi_i : X^{[n]} \rightarrow X^{[n-1]}$ on the simplicial manifold $X_\bullet = \{X_n = X^{[n+1]}\}$. We can define a bifunctor

$$\pi_i^* : \mathbf{BGrb}_{X^{[n-1]}} \longrightarrow \mathbf{BGrb}_{X^{[n]}}$$

sending objects, stable isomorphisms and 2-morphisms to the pull-backs by π_i ($i = 1, \dots, n$). One can then use this language to describe the data of a bundle 2-gerbe as follows. A bundle 2-gerbe on M consists of the data of a smooth surjective submersion $\pi : X \rightarrow M$ together with

- (1) An object $(\mathcal{Q}, Y, X^{[2]})$ in $\mathbf{BGrb}_{X^{[2]}}$.
- (2) A stable isomorphism $m : \pi_1^* \mathcal{Q} \otimes \pi_3^* \mathcal{Q} \rightarrow \pi_2^* \mathcal{Q}$ in $\mathbf{BGrb}_{X^{[3]}}$ defining the bundle 2-gerbe product which is associative up to a 2-morphism ϕ in $\mathbf{BGrb}_{X^{[4]}}$.
- (3) The 2-morphism ϕ satisfies a natural coherency condition in $\mathbf{BGrb}_{X^{[5]}}$.

We now briefly pause to describe some new notation which provides a good way to encode the simplicial bundle gerbe data (Cf. Definition 4.1 and Remark 4.2). We define maps $\pi_{ij} : X^{[n]} \rightarrow X^{[2]}$ for $n > 2$ which send a point (x_1, \dots, x_n) of $X^{[n]}$ to the point $(x_i, x_j) \in X^{[2]}$. It is clear that these maps can be written (non-uniquely) in terms of the π_i (the non-uniqueness stems from the simplicial identities satisfied by the face maps π_i 's). Let us write \mathcal{Q}_{ij} for $\pi_{ij}^* \mathcal{Q}$. For example, the bundle gerbe \mathcal{Q}_{12} over $X^{[3]}$ is the pull-back $\pi_3^* \mathcal{Q}$ of \mathcal{Q} .

Returning to the definition of bundle 2-gerbe, the next part of the definition requires that there is a stable isomorphism $m : \mathcal{Q}_{23} \otimes \mathcal{Q}_{12} \rightarrow \mathcal{Q}_{13}$ of bundle gerbes over $X^{[3]}$ together with a natural transformation called an associator

$$\phi : \pi_3^* m \circ (\pi_1^* m \otimes Id) \rightarrow \pi_2^* m \circ (Id \otimes \pi_4^* m)$$

which is a 2-morphism in the bi-category $\mathbf{BGrb}_{X^{[4]}}$, between the induced stable isomorphisms of bundle gerbes over $X^{[4]}$ making the following diagram commute up to an associator ϕ (represented by a 2-arrow in the diagram):

$$(4.2) \quad \begin{array}{ccc} \mathcal{Q}_{34} \otimes \mathcal{Q}_{23} \otimes \mathcal{Q}_{12} & \xrightarrow{Id \otimes \pi_4^* m} & \mathcal{Q}_{34} \otimes \mathcal{Q}_{13} \\ \pi_1^* m \otimes Id \downarrow & \nearrow \phi & \downarrow \pi_2^* m \\ \mathcal{Q}_{24} \otimes \mathcal{Q}_{12} & \xrightarrow{\pi_3^* m} & \mathcal{Q}_{14} \end{array}$$

Here we write $\pi_1^* m$ as a stable isomorphism $\mathcal{Q}_{34} \otimes \mathcal{Q}_{23} \rightarrow \mathcal{Q}_{24}$ over $X^{[4]}$, similarly for $\pi_2^* m, \pi_3^* m$ and $\pi_4^* m$. Hence, the associator ϕ as a 2-morphism in $\mathbf{BGrb}_{X^{[4]}}$ defines a line bundle \mathcal{L}_ϕ over $X^{[4]}$, which is required to have a trivialisation section.

In order to write the efficiently coherence condition satisfied by the natural transformation ϕ , we need one last piece of new notation. Let us write $\mathcal{Q}_{ijk} = \mathcal{Q}_{jk} \otimes \mathcal{Q}_{ij}$, $\mathcal{Q}_{ijkl} = \mathcal{Q}_{kl} \otimes \mathcal{Q}_{jk} \otimes \mathcal{Q}_{ij}$ and so on. So for example, $\mathcal{Q}_{123} = \mathcal{Q}_{23} \otimes \mathcal{Q}_{12}$, $\mathcal{Q}_{1234} =$

$\mathcal{Q}_{34} \otimes \mathcal{Q}_{23} \otimes \mathcal{Q}_{12}$ and the diagram (4.2) in $\mathbf{BGrb}_{X^{[4]}}$ can be written as

$$\begin{array}{ccc} \mathcal{Q}_{1234} & \xrightarrow{\quad} & \mathcal{Q}_{134} \\ \downarrow & \nearrow \phi & \downarrow \\ \mathcal{Q}_{124} & \xrightarrow{\quad} & \mathcal{Q}_{14} \end{array}$$

which is commutative if and only if ϕ is the identity 2-morphism denoted by Id .

The coherency condition satisfied by the natural transformation ϕ can then be viewed from the following two equivalent diagrams in $\mathbf{BGrb}_{X^{[5]}}$ calculating the associator (2-morphism) between the two induced stable isomorphisms from \mathcal{Q}_{12345} to \mathcal{Q}_{15} (one is $\mathcal{Q}_{12345} \rightarrow \mathcal{Q}_{1345} \rightarrow \mathcal{Q}_{145} \rightarrow \mathcal{Q}_{15}$, and the other is $\mathcal{Q}_{12345} \rightarrow \mathcal{Q}_{1235} \rightarrow \mathcal{Q}_{125} \rightarrow \mathcal{Q}_{15}$.)

$$(4.3) \quad \begin{array}{ccccc} & \mathcal{Q}_{12345} & \xrightarrow{\quad} & \mathcal{Q}_{1345} & \\ \mathcal{Q}_{1235} & \swarrow & \text{---} & \searrow & \mathcal{Q}_{145} \\ & \mathcal{Q}_{1245} & \xleftarrow{\pi_1^*\phi} & \xrightarrow{\pi_3^*\phi} & \\ & \mathcal{Q}_{125} & \xleftarrow{\pi_4^*\phi} & \xrightarrow{\pi_5^*\phi} & \mathcal{Q}_{15} \\ & \mathcal{Q}_{12345} & \xrightarrow{\quad} & \mathcal{Q}_{1345} & \\ & \mathcal{Q}_{1235} & \swarrow & \text{---} & \mathcal{Q}_{145} \\ & \mathcal{Q}_{135} & \xleftarrow{\pi_2^*\phi} & \xrightarrow{\quad} & \\ & \mathcal{Q}_{125} & \xleftarrow{\quad} & \xrightarrow{\quad} & \mathcal{Q}_{15} \end{array}$$

which implies the canonical isomorphism of two trivial line bundles over $X^{[5]}$

$$\pi_1^*\mathcal{L}_\phi \otimes \pi_3^*\mathcal{L}_\phi \otimes \pi_5^*\mathcal{L}_\phi \cong \pi_2^*\mathcal{L}_\phi \otimes \pi_4^*\mathcal{L}_\phi.$$

Remark 4.4. The first example of a bundle 2-gerbe is the tautological bundle 2-gerbe constructed in [14] over a 3-connected manifold M with a closed 4-form $\Theta \in \Omega_{cl,0}^4(M)$, see [44] for a detailed proof and more examples.

Definition 4.5. Let $(\mathcal{Q}, Y; X, M)$ be bundle 2-gerbe on M . A *bundle 2-gerbe connection* on \mathcal{Q} is a pair (∇, B) where ∇ is a bundle gerbe connection on the bundle gerbe $(\mathcal{Q}, Y; X^{[2]})$ and B is a curving for the bundle gerbe with connection (\mathcal{Q}, ∇) , whose bundle gerbe curvature ω on $X^{[2]}$ satisfies $\delta(\omega) = 0$, where $\delta = \pi_1^* - \pi_2^* + \pi_3^* : \Omega^*(X^{[2]}) \rightarrow \Omega^*(X^{[3]})$. Then we can solve the equation

$$\omega = (\pi_1^* - \pi_2^*)(C)$$

for a three form C on X , such a choice of C is called a *2-curving* for the bundle 2-gerbe $(\mathcal{Q}, Y; X, M)$, or simply the *bundle 2-gerbe curving*. Then $dC = \pi^*(\Theta)$ for a closed four form Θ on M , which is called the *bundle 2-gerbe curvature*.

Locally, as in [44] and [34], a bundle 2-gerbe on M with connection and curving is determined by a degree 3 Deligne class

$$(4.4) \quad [(g_{ijkl}, A_{ijk}, B_{ij}, C_i)] \in H^3(M, \mathcal{D}^3)$$

for a good cover $\{U_i\}$ of M , over which there are local sections $s_i : U_i \rightarrow X$. Then $C_i = s_i^*C$. Using (s_i, s_j) , we can pull-back the bundle gerbe $(\mathcal{Q}, Y, X^{[2]})$ to $U_{ij} = U_i \cap U_j$, such that $\mathcal{Q}_{ij} = (s_i, s_j)^*\mathcal{Q}$ is trivial. Then the bundle 2-gerbe product gives rise to the following stable isomorphism of bundle gerbes with connection and curving:

$$\mathcal{Q}_{ij} \otimes \mathcal{Q}_{jk} \rightarrow \mathcal{Q}_{ik} \otimes \delta(\mathcal{G}_{ijk})$$

for a bundle gerbe \mathcal{G}_{ijk} with connection and curving over $U_{ijk} = U_i \cap U_j \cap U_k$, hence the curving $B_{ij} = (s_i, s_j)^*B$ satisfies

$$dB_{ij} = C_i - C_j, \quad B_{ij} + B_{jk} + B_{ki} = dA_{ijk}$$

for a connection 1-form A_{ijk} on \mathcal{G}_{ijk} . Moreover, the associator ϕ defines a $U(1)$ -valued function g_{ijkl} over U_{ijkl} such that g_{ijkl} satisfies the Čech 3-cocycle condition

$$g_{ijkl}g_{ijkm}^{-1}g_{ijlm}g_{iklm}^{-1}g_{jklm} = 1$$

and

$$A_{ijk} - A_{ijl} + A_{ikl} - A_{jkl} = g_{ijkl}^{-1}dg_{ijkl}.$$

Definition 4.6. A bundle 2-gerbe $(\mathcal{Q}, Y; X, M)$ is called trivial if $(\mathcal{Q}, Y; X^{[2]})$ is isomorphic in $\mathbf{BGrb}_{X^{[2]}}$ to

$$\delta(\mathcal{G}) = \pi_2^*(\mathcal{G}^*) \otimes \pi_1^*(\mathcal{G})$$

for a bundle gerbe \mathcal{G} over X together with compatible conditions on bundle 2-gerbe products in $\mathbf{BGrb}_{X^{[3]}}$ and associator natural transformations in $\mathbf{BGrb}_{X^{[4]}}$ (see [44] for more details). A bundle 2-gerbe $(\mathcal{Q}_1, Y_1; X, M)$ is called stably isomorphic to a bundle 2-gerbe $(\mathcal{Q}_2, Y_2; X, M)$ if and only if \mathcal{Q}_1 is isomorphic to $\mathcal{Q}_2 \otimes \delta(\mathcal{G})$ for a bundle gerbe \mathcal{G} over X together with extra conditions involving the associator natural isomorphisms for \mathcal{Q}_1 and \mathcal{Q}_2 .

Lemma 4.7. *Let $(\mathcal{P}, X; Y, M)$ be a bundle 2-gerbe with connection and curving. Suppose there exists a stable isomorphism of bundle gerbes $(\mathcal{P}, X) \cong (\mathcal{Q}, Z)$ over $Y^{[2]}$. Then there exists a bundle 2-gerbe structure (\mathcal{Q}, Z, Y, M) with induced connection and curving which has the same Deligne class in $H^3(M, \mathcal{D}^3)$ as the original bundle 2-gerbe $(\mathcal{P}, X; Y, M)$.*

Proof. First we must show that $(\mathcal{Q}, Z; Y, M)$ admits a bundle 2-gerbe product. We use the bundle 2-gerbe product on $(\mathcal{P}, X; Y, M)$ to define it. Recall that this product is a stable isomorphism of bundle gerbes over $Y^{[3]}$,

$$\pi_1^*\mathcal{P} \otimes \pi_3^*\mathcal{P} \cong \pi_2^*\mathcal{P}$$

It is convenient here to realise the stable isomorphism as a trivial bundle gerbe by expressing the product as a bundle gerbe isomorphism

$$\pi_1^*\mathcal{P} \otimes \pi_2^*\mathcal{P}^* \otimes \pi_3^*\mathcal{P} = \delta(J)$$

Similarly we have an isomorphism

$$\mathcal{P} = \mathcal{Q} \otimes \delta(L)$$

representing the stable isomorphism of bundle gerbes over $Y^{[2]}$. Thus we have

$$\pi_1^*(\mathcal{Q} \otimes \delta(L)) \otimes \pi_2^*(\mathcal{Q} \otimes \delta(L))^* \otimes \pi_3^*(\mathcal{Q} \otimes \delta(L)) = \delta(J)$$

and so

$$\pi_1^*\mathcal{Q} \otimes \pi_2^*\mathcal{Q}^* \otimes \pi_3^*\mathcal{Q} = \delta(J) \otimes \pi_1^*\delta(L)^* \otimes \pi_2^*\delta(L) \otimes \pi_3^*\delta(L)^*$$

where we use the fact that $\pi_i^*\delta(L) \otimes \pi_i^*\delta(L)^*$ is canonically trivial. Since the pullback of a trivial bundle gerbe must itself be trivial and a tensor product of trivial bundle gerbes is trivial then the right hand side is trivial and thus we potentially have a bundle 2-gerbe product for \mathcal{Q} . To confirm that it does define a bundle 2-gerbe product we must check the associativity conditions.

Note that it is helpful now to look at diagram (4.2) and diagram (4.3) to understand the following arguments. Recall that there is a bundle called the associator bundle on $X^{[4]}$ which is the obstruction to the bundle 2-gerbe product being associative. It can be defined by considering the product

$$\pi_1^{-1}\delta(J) \otimes \pi_2^{-1}\delta(J)^* \otimes \pi_3^{-1}\delta(J) \otimes \pi_4^{-1}\delta(J)^*$$

where $\pi_i : Y^{[4]} \rightarrow Y^{[3]}$ are the face maps in the simplicial complex. This product defines the associator line bundle on $Y^{[4]}$ (Cf. Remark 4.2 and Diagram (4.2)). Changing to the trivialisation representing the bundle gerbe product for \mathcal{Q} , we find that the extra terms involving $\delta(J)$ all cancel (in the sense of having canonical trivialisations), hence the associator line bundles for \mathcal{P} and \mathcal{Q} are the same, so the bundle 2-gerbe product for \mathcal{Q} is well defined. With the induced connection and curving, it is straight forward to show that the Deligne class is cohomologous to the the Deligne class for $(\mathcal{P}, X; Y, M)$. \square

From Lemma 4.7, we know that two stably isomorphic bundle 2-gerbes with connection and curving have the same Deligne class. Given a representative of a Deligne class as in (4.4), we can construct a local bundle 2-gerbe with connection and curving over M as in [44] and [34]. Analogous to the fact that $H^2(M, \mathcal{D}^2)$ classifies stable equivalence classes of bundle gerbes with connection and curving, we have the following proposition, whose complete proof can be found in [34].

Proposition 4.8. (Cf. [34]) *The group of stable isomorphism classes of bundle 2-gerbes with connection and curving over M is isomorphic to $H^3(M, \mathcal{D}^3)$.*

5. MULTIPLICATIVE BUNDLE GERBES

The simplicial manifold BG_\bullet associated to the classifying space of G is constructed in [19], where the total space of the universal G -bundle EG also has a simplicial manifold structure. The simplicial manifold

$$BG_\bullet = \{BG_n = G \times \cdots \times G \text{ (n copies)}\}$$

(where $n = 0, 1, 2, \dots$), is endowed with face operators $d_i : G^{n+1} \rightarrow G^n$, ($i = 0, 1, \dots, n+1$)

$$d_i(g_0, \dots, g_n) = \begin{cases} (g_1, \dots, g_n), & i = 0, \\ (g_1, \dots, g_{i-1}g_i, g_{i+1}, \dots, g_n), & 1 \leq i \leq n, \\ (g_0, \dots, g_{n-1}), & i = n+1. \end{cases}$$

Definition 5.1. A multiplicative bundle gerbe over a compact Lie group G is defined to be a simplicial bundle gerbe on the simplicial manifold BG_\bullet associated to the classifying space of G .

For a compact, simply connected, simple Lie group G , the tautological bundle over G associated to any class in $H^3(G; \mathbb{Z})$ is a simplicial bundle gerbe as shown in [44], hence, a multiplicative bundle gerbe.

Proposition 5.2. *Let G be a compact, connected Lie group. Then there is an isomorphism between $H^4(BG; \mathbb{Z})$ and the space of isomorphism classes of multiplicative bundle gerbes on G .*

First of all, it is not very hard to see that $H^4(BG; \mathbb{Z})$ corresponds to isomorphism classes of simplicial bundle gerbes on the simplicial manifold $EG^{[\bullet]}$. This is because a simplicial bundle gerbe on $EG^{[\bullet]}$ is the same thing as a bundle 2-gerbe on $EG \rightarrow BG$. Here we say that two simplicial bundle gerbes \mathcal{G} and \mathcal{Q} on $EG^{[\bullet]}$ are isomorphic if there is a stable isomorphism $\mathcal{G} \cong \mathcal{Q}$ which is compatible with all the multiplicative structures on \mathcal{G} and \mathcal{Q} . On a general simplicial manifold X_\bullet the notion of isomorphism of simplicial bundle gerbes is more complicated, involving bundle gerbes on X_0 , however because EG is contractible we may use this simpler notion of isomorphism without any loss of generality.

Recall from [7] and [10] the definition of the simplicial Čech cohomology groups $H^*(X_\bullet; \underline{A})$ for a simplicial manifold X_\bullet and some topological abelian group A . To define these one first needs the notion of a covering of the simplicial manifold X_\bullet . By definition this is a family of covers $\mathcal{U}^n = \{U_\alpha^n\}$ of the manifolds X_n which are compatible with the face and degeneracy operators for X_\bullet . Brylinski and McLaughlin in [10] explain how one may inductively construct such a family of coverings by first starting with an arbitrary cover \mathcal{U}^0 of X_0 and then choosing a common refinement \mathcal{U}^1 of the induced covers $d_0^{-1}(\mathcal{U}^0)$ and $d_1^{-1}(\mathcal{U}^0)$ of X_1 . \mathcal{U}^1 then induces three covers $d_0^{-1}(\mathcal{U}^1)$, $d_1^{-1}(\mathcal{U}^1)$ and $d_2^{-1}(\mathcal{U}^1)$ of X_2 . One then chooses a common refinement \mathcal{U}^2 and repeats this process. In particular, the covering \mathcal{U}^\bullet may be chosen so that each \mathcal{U}^n is a good cover of X_n .

The simplicial Čech cohomology $H^*(\mathcal{U}^\bullet, \underline{A})$ of X_\bullet for the covering \mathcal{U}^\bullet is by definition the cohomology of the double complex $C^p(\mathcal{U}^q, \underline{A})$ where $C^*(\mathcal{U}^p, \underline{A})$ is the Čech complex for the covering \mathcal{U}^p of the manifold X_p . The differential for the complex $C^p(\mathcal{U}^q, \underline{A})$ is induced in the usual way from the face operators d_i on X_\bullet . The groups $H^*(X_\bullet, \underline{A})$ are then defined by taking a direct limit of the coverings \mathcal{U}^\bullet . If the covering \mathcal{U}^\bullet is good in the sense that each \mathcal{U}^n is a good cover then

$$H^*(X_\bullet; \underline{A}) \cong H^*(\mathcal{U}^\bullet; \underline{A}).$$

Note that there is a spectral sequence converging to a graded quotient of $H^*(X_\bullet; \underline{A})$ with

$$E_1^{p,q} = H^p(X_q; \underline{A}).$$

The following proposition is a straightforward extension of Theorem 5.7 in part I of [10] to the language of bundle gerbes.

Proposition 5.3. *Let X_\bullet be a simplicial manifold. Then we have that isomorphism classes of simplicial bundle gerbes on X_\bullet are classified by the simplicial Čech cohomology group $H^3(X_{\bullet \geq 1}; \underline{U}(1))$. Here $X_{\bullet \geq 1}$ denotes the truncation of X_\bullet through degrees ≥ 1 .*

We sketch a proof of this Proposition below. Let us first be clear about what we mean by the group $H^3(X_{\bullet \geq 1}; \underline{\mathbb{C}}^*)$. By this we mean that if \mathcal{U}^\bullet is a good covering of X_\bullet , so that $H^*(X_\bullet, \underline{U}(1)) = H^*(\mathcal{U}^\bullet, \underline{U}(1))$, then $H^*(X_{\bullet \geq 1}, \underline{U}(1))$ is the cohomology

of the double complex $C^p(\mathcal{U}^q; \underline{U}(1))$ with q in degrees ≥ 1 . Also, by an isomorphism of simplicial bundle gerbes on X_\bullet we mean a stable isomorphism $\mathcal{G} \cong \mathcal{Q}$ compatible with all the product structures on \mathcal{G} and \mathcal{Q} . We do not require that $\mathcal{G} \cong \mathcal{Q} \otimes \delta(T)$ for some bundle gerbe T on X_0 . As noted above this is a restrictive definition but is sufficient for the cases we are interested in, such as the contractible space EG .

Given a simplicial bundle gerbe \mathcal{G} on X_\bullet , we associate to \mathcal{G} a simplicial Čech cohomology class in $H^3(X_{\bullet \geq 1}, \underline{U}(1))$ as follows. For the good covering \mathcal{U}^\bullet on X_\bullet , let $\underline{g} = (g_{\alpha\beta\gamma})$ be a Čech cocycle representative for the Dixmier-Douady class of \mathcal{G} . Then it is easy to see that the 2-cocycle

$$d_0^*(\underline{g}) d_1^*(\underline{g}^{-1}) d_2^*(\underline{g}) = \delta(\underline{h})$$

for some 1-cochain $\underline{h} = (h_{\alpha\beta})$ on the covering \mathcal{U}^2 . Then the simplicial Čech 1-cochain

$$d_0^*(\underline{h}) d_1^*(\underline{h}^{-1}) d_2^*(\underline{h}) d_3^*(\underline{h}^{-1})$$

on the cover \mathcal{U}^3 is a 1-cocycle: it is a Čech representative for the first Chern class of the associator line bundle on X_3 (the line bundle induced by the associator). Consequently, we must have

$$d_0^*(\underline{h}) d_1^*(\underline{h}^{-1}) d_2^*(\underline{h}) d_3^*(\underline{h}^{-1}) = \delta(\underline{k})$$

for some simplicial Čech 0-cochain $\underline{k} = (k_\alpha)$ on \mathcal{U}^3 . The cocycle condition for the associator section shows that we must have

$$d_0^*(\underline{k}) d_1^*(\underline{k}^{-1}) d_2^*(\underline{k}) d_3^*(\underline{k}^{-1}) d_4^*(\underline{k}) = 1$$

on \mathcal{U}^4 . The triple $(\underline{g}, \underline{h}, \underline{k})$ is a simplicial Čech cocycle in the truncated double complex $C^p(\mathcal{U}^{q \geq 1}, \underline{U}(1))$ representing a class in $H^3(X_{\bullet \geq 1}, \underline{U}(1))$. Conversely, such a simplicial Čech cocycle $(\underline{d}, \underline{h}, \underline{k})$ in the truncated double complex $C^p(\mathcal{U}^{q \geq 1}, \underline{U}(1))$ determines a unique isomorphism class of simplicial bundle gerbes on X_\bullet . If the cocycle $(\underline{d}, \underline{h}, \underline{k})$ associated to a simplicial bundle gerbe \mathcal{G} is trivial in $H^3(X_{\bullet \geq 1}, \underline{U}(1))$, then there is a stable isomorphism between \mathcal{G} and the trivial simplicial bundle gerbe, consisting of a trivial bundle gerbe on X_1 equipped with the trivial product structures.

As an immediate consequence of Proposition 5.3, we see that isomorphism classes of simplicial bundle gerbes on BG_\bullet are classified by the simplicial Čech cohomology group $H^3(BG_\bullet, \underline{U}(1))$. Since the simplicial map $EG^{[\bullet]} \rightarrow BG_\bullet$ is a homotopy equivalence in each degree, we see from the spectral sequence above that it induces an isomorphism

$$H^3(BG_\bullet, \underline{U}(1)) \cong H^3(EG^{[\bullet]}, \underline{U}(1)).$$

As we have already noted, isomorphism classes of simplicial bundle gerbes on $EG^{[\bullet]}$ correspond exactly to isomorphism classes of bundle 2-gerbes on $EG \rightarrow BG$ and hence to $H^4(BG; \mathbb{Z})$.

Given a principal G -bundle $\pi : P \rightarrow M$, there exists a natural map

$$\hat{g} : P^{[2]} \longrightarrow G$$

given by $p_1 \cdot \hat{g}(p_1, p_2) = p_2$ such that $P^{[2]} \rightarrow P \times G$ sending (p_1, p_2) to $(p_1, \hat{g}(p_1, p_2))$ is a diffeomorphism.

Lemma 5.4. *Given a principal G -bundle $P \rightarrow M$ and a multiplicative bundle gerbe \mathcal{G} over G then there exists a bundle 2-gerbe over M of the form $(\mathcal{Q}, X; P, M)$ such that $(\mathcal{Q}, X, P^{[2]})$ is the pull-back bundle gerbe $\hat{g}^*\mathcal{G}$.*

Proof. Note that there exists a diffeomorphism

$$P \times G^n \longrightarrow P^{[n+1]}$$

given by

$$(p, g_1, g_2, \dots, g_n) \mapsto (p, p \cdot g_1, p \cdot g_1 g_2, \dots, p \cdot (g_1 g_2 \cdots g_n)).$$

The inverse of this map, composing with the projection to G^n , defines to a simplicial map

$$\hat{g}_\bullet : P_\bullet = \{P_n = P^{[n+1]}\} \longrightarrow BG_\bullet = \{BG_n = G^n\},$$

which can be used to pull back the simplicial bundle gerbe over BG_\bullet corresponding to \mathcal{G} to a simplicial bundle gerbe $\tilde{\mathcal{Q}}$ over P_\bullet . This defines a bundle 2-gerbe $(\tilde{\mathcal{Q}}, Y; P, M)$ over M of the required form. \square

Given a multiplicative bundle gerbe \mathcal{G} over G , applying the above Lemma 5.4 to the universal bundle $\pi : EG \rightarrow BG$, we obtain a bundle 2-gerbe over BG of the form $(\hat{g}^* \mathcal{G}; EG, BG)$ with $\hat{g}^* \mathcal{G}$ is a bundle gerbe over $EG^{[2]}$ obtained by the pull-back of \mathcal{G} via $\hat{g} : EG^{[2]} \rightarrow G$. Conversely, we will show that every bundle 2-gerbe over BG is stably isomorphic a bundle 2-gerbe of this form.

Lemma 5.5. *Every bundle 2-gerbe on BG is stably isomorphic to a bundle 2-gerbe of the form $(\tilde{\mathcal{Q}}, X; EG, BG)$, where $(\tilde{\mathcal{Q}}, X)$ is a bundle gerbe over $EG^{[2]}$.*

Proof. We use the classifying theory of bundle 2-gerbes. It is well known that $H^4(M, \mathbb{Z}) \cong [M; K(\mathbb{Z}, 4)]$. We use the iterated classifying space $BBBU(1)$ (or $B^3U(1)$) as a model for $K(\mathbb{Z}, 4)$ with a differential space structure constructed in [26]. In [26] Theorem H , it is shown that for a smooth manifold M , the group $H^4(M, \mathbb{Z})$ is isomorphic to the group of isomorphism classes of smooth principal $B^2U(1)$ -bundles over M .

We can transgress the degree 4 class in $H^4(B^3U(1), \mathbb{Z})$ to get a degree 3 class in $H^3(B^2U(1), \mathbb{Z})$ which determines a multiplicative bundle gerbe over $B^2U(1)$. Then we apply the canonical map $\hat{g} : EB^2U(1)^{[2]} \rightarrow B^2U(1)$ to pull-back the corresponding multiplicative bundle gerbe over $B^2U(1)$ to $EB^2U(1)^{[2]}$. This gives rise to the universal bundle 2-gerbe $\tilde{\mathcal{Q}}$ over $B^3U(1)$. The classifying bundle 2-gerbe then has the form

$$\begin{array}{ccc} \tilde{\mathcal{Q}} & & \\ \downarrow & & \\ EB^2U(1)^{[2]} & \Rightarrow & EB^2U(1) \\ & & \downarrow \\ & & B^3U(1) \end{array}$$

As $B^3U(1)$ is 3-connected, and $H^4(B^3U(1), \mathbb{Z}) \cong \mathbb{Z}$, the tautological bundle 2-gerbe developed in [14] can be adapted to give another construction of such a classifying bundle 2-gerbe over $B^3U(1)$ associated to any integral class in $H^4(B^3U(1), \mathbb{Z})$.

Any bundle 2-gerbe on BG is defined by pulling back the universal bundle 2-gerbe by a classifying map $\psi : BG \rightarrow B^3U(1)$. Consider the map $\pi \circ \psi : EG \rightarrow B^3U(1)$ where π is the projection in the universal G -bundle. By using the homotopy lifting property and the contractibility of EG we can always find a lift $\hat{\psi}$,

$$\begin{array}{ccc} EG & \xrightarrow{\hat{\psi}} & EB^2U(1) \\ \downarrow & & \downarrow \\ BG & \xrightarrow{\psi} & B^3U(1) \end{array}$$

Thus we can pull back the universal bundle 2-gerbe to get a bundle 2-gerbe of the form $(\mathcal{Q}, X; EG, BG)$, with (\mathcal{Q}, X) a bundle gerbe over $EG^{[2]}$. \square

Lemma 5.6. *Any bundle gerbe over $EG^{[2]}$ is stably isomorphic to $\hat{g}^*\mathcal{G}$ where $\hat{g} : EG^{[2]} \rightarrow G$ is the map satisfying $e_2 = e_1\hat{g}(e_1, e_2)$ for $(e_1, e_2) \in EG^{[2]}$ and \mathcal{G} is some bundle gerbe over G .*

Proof. We may identify $EG^{[2]}$ with $EG \times G$ via the map $(e_1, e_2) \mapsto (e_1, \hat{g}(e_1, e_2))$. Thus stable isomorphism classes of bundle gerbes over $EG^{[2]}$ are classified by $H^3(EG \times G, \mathbb{Z})$ which, since EG is contractible, is equal to $H^3(G; \mathbb{Z})$. Under the identification we see that the Dixmier-Douady class of the bundle gerbe on $EG^{[2]}$ must be obtained from a class in $H^3(G, \mathbb{Z})$ by the map \hat{g} . \square

Proposition 5.7. *Every bundle 2-gerbe on BG is stably isomorphic to a bundle 2-gerbe of the form $(\hat{g}^*\mathcal{G}, X; EG, BG)$ for a multiplicative bundle gerbe \mathcal{G} over G .*

Proof. We start with any bundle 2-gerbe on BG . By Lemma 5.5 we may assume without loss of generality that it is of the form $(\mathcal{Q}, X; EG, BG)$. Next we use lemma 5.6 to replace the bundle gerbe $(\mathcal{Q}, X, EG^{[2]})$ with $(\hat{g}^*\mathcal{G}, \hat{g}^*Y, EG^{[2]})$ where (\mathcal{G}, Y, G) is now a bundle gerbe over G . Note that $(\mathcal{Q}, X; EG^{[2]})$ and $(\hat{g}^*\mathcal{G}, \hat{g}^*Y, EG^{[2]})$ is stably isomorphic. Using Lemma 4.7, we know that there exists a bundle 2-gerbe structure on $(\hat{g}^*\mathcal{G}, \hat{g}^*Y; EG, BG)$ which does not change the stable isomorphism class of the original bundle 2-gerbe on BG . For the bundle 2-gerbe $(\hat{g}^*\mathcal{G}, \hat{g}^*Y; EG, BG)$ to be defined, $\hat{g}^*\mathcal{G}$ must be a simplicial bundle gerbe over $EG_\bullet = \{EG_n = EG^{[n+1]}\}$. Fix a point $p \in EG$, then $\hat{p} : G^n \rightarrow EG^{[n+1]}$ with

$$\hat{p}(g_1, g_2, \dots, g_n) = (p, p \cdot g_1, p \cdot g_1g_2, \dots, p \cdot (g_1g_2 \cdots g_n)),$$

is a simplicial map between $BG_\bullet = \{BG_n = G^n\}$ and EG_\bullet . Then $\mathcal{G} \cong \hat{p}^* \circ \hat{g}^*\mathcal{G}$ implies that \mathcal{G} is a simplicial bundle gerbe over BG_\bullet . Hence, \mathcal{G} is a multiplicative bundle gerbe over G . \square

Theorem 5.8. *Given a bundle gerbe \mathcal{G} over G , \mathcal{G} is multiplicative if and only if its Dixmier-Douady class is transgressive, i.e., in the image of the transgression map $\tau : H^4(BG, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z})$.*

Proof. Given a multiplicative bundle gerbe \mathcal{G} over G , applying Lemma 5.4 to the universal bundle $\pi : EG \rightarrow BG$, we obtain a bundle 2-gerbe \mathcal{Q} over BG of the form $(\hat{g}^*\mathcal{G}; EG, BG)$ with $\hat{g}^*\mathcal{G}$ is a bundle gerbe over $EG^{[2]}$ obtained by the pull-back of \mathcal{G} via $\hat{g} : EG^{[2]} \rightarrow G$. The stable isomorphism class of $(\hat{g}^*\mathcal{G}; EG, BG)$ defines a class $\phi \in H^4(BG, \mathbb{Z})$. We now show that the transgression of ϕ under the map $\tau : H^4(BG, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z})$ is the Dixmier-Douady class of \mathcal{G} , hence transgressive.

The class ϕ defines a homotopy class in $[BG, K(\mathbb{Z}, 4)]$ such that the bundle 2-gerbe \mathcal{Q} is stably isomorphic to a bundle 2-gerbe obtained by pulling back the universal bundle 2-gerbe $\tilde{\mathcal{Q}}$ over $B^3U(1) = K(\mathbb{Z}, 4)$ via a classifying map $\psi : BG \rightarrow K(\mathbb{Z}, 4)$ of ϕ and the commutative diagram

$$\begin{array}{ccc} EG & \xrightarrow{\hat{\psi}} & EK(\mathbb{Z}, 3) \\ \downarrow & & \downarrow \\ BG & \xrightarrow{\psi} & K(\mathbb{Z}, 4). \end{array}$$

This commutative diagram gives rise to a homotopy class of maps $\hat{\psi} : G \rightarrow K(\mathbb{Z}, 3)$ which determines a class

$$c_\psi \in H^3(G, \mathbb{Z}) \cong [G, K(\mathbb{Z}, 3)].$$

Using the long exact sequence for homotopy groups for the $K(\mathbb{Z}, 3)$ -fibration $EK(\mathbb{Z}, 3) \rightarrow BK(\mathbb{Z}, 3) = K(\mathbb{Z}, 4)$ and the Hurewicz theorem for 3-connected spaces, we know that the transgression map $H^4(BK(\mathbb{Z}, 3), \mathbb{Z}) \rightarrow H^3(K(\mathbb{Z}, 3), \mathbb{Z})$ sends the generator of $H^4(BK(\mathbb{Z}, 3), \mathbb{Z})$ to the generator of $H^3(K(\mathbb{Z}, 3), \mathbb{Z})$. Therefore, we obtain that

$$c_\psi = \tau(\phi),$$

as $\phi \in H^4(BG, \mathbb{Z}) \cong [BG, K(\mathbb{Z}, 4)]$ is defined by pulling back the generator of $H^4(BK(\mathbb{Z}, 3), \mathbb{Z})$ via the classifying map ψ .

As bundle gerbes over $EG^{[2]}$, $\mathcal{Q} = \hat{g}^*\mathcal{G}$ is stably isomorphic to the pull-back bundle gerbe $(\hat{\psi}^{[2]})^*\tilde{\mathcal{Q}}$ via the map $\hat{\psi}^{[2]} : EG^{[2]} \rightarrow EK(\mathbb{Z}, 3)^{[2]}$. This implies that the Dixmier-Douady class of $\hat{g}^*\mathcal{G}$ is given by the homotopy class of the map

$$EG^{[2]} \xrightarrow{\hat{\psi}^{[2]}} EK(\mathbb{Z}, 3)^{[2]} \longrightarrow K(\mathbb{Z}, 3).$$

As \hat{g}^* induces an isomorphism $H^3(G, \mathbb{Z}) \rightarrow H^3(EG^{[2]}, \mathbb{Z})$, putting all these together, we know that the Dixmier-Douady class of \mathcal{G} is given by $c_\psi = \tau(\phi)$, hence, transgressive.

Conversely, suppose \mathcal{G} is a bundle gerbe over G whose Dixmier-Douady class is a transgressive class $\tau(c) \in H^3(G, \mathbb{Z})$ for a class $c \in H^4(BG, \mathbb{Z})$. By Proposition 5.7 we may, without loss of generality, realise c by a bundle 2-gerbe $(\mathcal{Q}_c, X; EG, BG)$ over BG from a multiplicative bundle gerbe \mathcal{G}_c over G such that $\mathcal{Q}_c = \hat{g}^*\mathcal{G}_c$. The above argument shows that the Dixmier-Douady class of \mathcal{G}_c is also given by $\tau(c)$. Therefore, the bundle gerbe \mathcal{G} is stably isomorphic to the multiplicative bundle gerbe \mathcal{G}_c . Then the multiplicative structure on \mathcal{G}_c

$$m_c : d_0^*\mathcal{G}_c \otimes d_2^*\mathcal{G}_c \rightarrow d_1^*\mathcal{G}_c$$

induces a multiplicative structure on \mathcal{G}

$$m : d_0^*\mathcal{G} \otimes d_2^*\mathcal{G} \rightarrow d_1^*\mathcal{G}$$

as a stable isomorphism in the bi-category \mathbf{BGrb}_{G^2} . The associator for $(\mathcal{G}, m_{\mathcal{G}})$ (see (4.2))

$$\phi : d_2^*m \circ (d_0^*m \otimes Id) \rightarrow d_1^*m \circ (Id \otimes d_3^*m)$$

is also induced by the corresponding associator ϕ_c for (\mathcal{G}_c, m_c) in the bi-category \mathbf{BGrb}_{G^3} . The coherent condition for ϕ (see (4.3)) follows from the corresponding coherent condition for ϕ_c in the bi-category \mathbf{BGrb}_{G^4} . Hence, the bundle gerbe \mathcal{G} over G , whose Dixmier-Douady class is transgressive, is multiplicative. \square

Note that there exists a simpler proof of Theorem 5.8 by applying Proposition 5.2 and an observation that $H^4(BG, \mathbb{Z})$ classifies the isomorphism classes of simplicial bundle gerbe over BG_\bullet . The constructive proof for Theorem 5.8 is given so that the proof can be adopted to establish the following theorem, which is the refinement of Theorem 5.8 for the correspondence from three dimensional Chern-Simons gauge theories with gauge group G to Wess-Zumino-Witten models on the group manifold G . The proof will be postponed until after we discuss Chern-Simons bundle 2-gerbes.

Theorem 5.9. Denote by $\Psi : CS(G) \longrightarrow WZW(G) \cong H^2(G, \mathcal{D}^2)$ the correspondence map and let \mathcal{G} be a bundle gerbe over G with connection and curving, whose Deligne class $d(\mathcal{G})$ is in $H^2(G, \mathcal{D}^2)$. Then $d(\mathcal{G}) \in Im(\Psi)$, if and only if \mathcal{G} is multiplicative.

Remark 5.10. Note that the transgression map $H^4(BSO(3), \mathbb{Z}) \rightarrow H^3(SO(3), \mathbb{Z})$ sends the generator of $H^4(BSO(3), \mathbb{Z})$ to twice of the generator of $H^3(SO(3), \mathbb{Z})$. Hence, only those bundle gerbes \mathcal{G}_{2k} over $SO(3)$, with even Dixmier-Douady classes in $H^3(SO(3), \mathbb{Z}) \cong \mathbb{Z}$, are multiplicative.

6. THE CHERN-SIMONS BUNDLE 2-GERBE

In this Section, we will construct a Chern-Simons bundle 2-gerbe over BG such that the proof of the main theorem (Theorem 5.9) follows.

Given a principal G -bundle P with a connection A over M , a Chern-Simons gauge theory $c \in CS(G)$ defines a degree 3 Deligne characteristic class

$$c(P, A) \in H^3(M, \mathcal{D}^3).$$

We will construct a bundle 2-gerbe with connection and curving over M corresponding to the Deligne class $c(P, A)$. We shall call this a *Chern-Simons bundle 2-gerbe*. We define it in terms of a universal Deligne characteristic class represented by the *universal Chern-Simons bundle 2-gerbe*.

In order to use differential forms on BG we need to fix a smooth infinite dimensional model of BG by embedding G into $U(N)$ and letting EG be the Stiefel manifold of N orthonormal vectors in a separable complex Hilbert space. Alternatively, we could choose a smooth finite dimensional n -connected approximation B_n of the classifying space BG as in [40]. Given a principal G -bundle P with a connection A over M and an integer $n > \max\{5, \dim M\}$, there is a choice of n -connected finite dimensional principal G -bundle E_n with a connection \bar{A} such that (E_n, \bar{A}) is a classifying space of (P, A) . In particular $H^k(B, \mathbb{Z}) \cong H^k(BG, \mathbb{Z})$ for $k \leq n$. For convenience, we suppress this latter detail and work directly on the infinite dimensional smooth model of $EG \rightarrow BG$.

Note that there exists a connection \mathbb{A} on the universal bundle EG over BG and a classifying map f for (P, A) such that

$$(P, A) = f^*(EG, \mathbb{A}),$$

which implies $c(P, A) = f^*c(EG, \mathbb{A})$ with $c(EG, \mathbb{A}) \in H^3(BG, \mathcal{D}^3)$ is the Deligne characteristic class for (EG, \mathbb{A}) . From the commutative diagram:

$$\begin{array}{ccc} H^3(BG, \mathcal{D}^3) & \longrightarrow & H^2(G, \mathcal{D}^2) \\ \downarrow^c & & \downarrow^c \\ H^4(BG, \mathbb{Z}) & \rightarrow & H^3(G, \mathbb{Z}), \end{array}$$

where the vertical maps are the characteristic class maps on Deligne cohomology groups, we see that Theorem 5.9 refines the result in Theorem 5.8.

Given a class $\phi \in H^4(BG, \mathbb{Z})$, denote by Φ the corresponding G -invariant polynomial on the Lie algebra of G . Then associated to a connection A on a principal G -bundle (with curvature F_A) there is a closed 4-form

$$\Phi\left(\frac{i}{2\pi}F_A\right) \in \Omega_{cl,0}^4(M)$$

with integer periods. In fact, $\Phi\left(\frac{i}{2\pi}F_A\right) = f^*\Phi\left(\frac{i}{2\pi}F_{\mathbb{A}}\right)$, and

$$\left(\phi, \Phi\left(\frac{i}{2\pi}F_{\mathbb{A}}\right)\right) \in H^4(BG, \mathbb{Z}) \times \Omega_{cl,0}^4(BG)$$

determines a unique Deligne class

$$c_\phi(EG, \mathbb{A}) \in H^3(BG, \mathcal{D}^3).$$

Hence $c_\phi(P, A) = f^*(c_\phi(EG, \mathbb{A})) \in H^3(M, \mathcal{D}^3)$. So we can say that a class $\phi \in H^4(BG, \mathbb{Z})$ defines a *canonical Chern-Simons gauge theory* c_ϕ with gauge group G .

From the exact sequence for $H^3(BG, \mathcal{D}^3)$,

$$0 \rightarrow \Omega^3(BG)/\Omega_{cl,0}^3(BG) \rightarrow H^3(BG, \mathcal{D}^3) \rightarrow H^4(BG, \mathbb{Z}) \rightarrow 0,$$

any Chern-Simons gauge theory with the same characteristic class ϕ in $H^4(BG, \mathbb{Z})$ differs from c_ϕ by a Deligne class $[1, 0, 0, C]$ for a 3-form C on BG . Note that $[1, 0, 0, C]$ defines a trivial bundle 2-gerbe over BG , so in this section, we only construct the universal Chern-Simons bundle 2-gerbe over BG corresponding to the canonical Chern-Simons gauge theory c_ϕ with gauge group G . Notice that for two different connections \mathbb{A}_1 and \mathbb{A}_2 on the universal bundle EG ,

$$c_\phi(EG, \mathbb{A}_1) - c_\phi(EG, \mathbb{A}_2) = [(1, 0, 0, CS_\phi(\mathbb{A}_1, \mathbb{A}_2))],$$

where $CS_\phi(\mathbb{A}_1, \mathbb{A}_2)$ is the Chern-Simons form on BG associated to a pair of connections \mathbb{A}_1 and \mathbb{A}_2 on EG (Cf. [17]).

Remark 6.1. Recall [16] that associated with each principal G -bundle P with connection A Cheeger and Simons constructed a differential character

$$S_{\Phi, \phi}(P, A) \in \check{H}^3(M, U(1)),$$

where $\Phi \in I^2(G)$ is a G -invariant polynomial on its Lie algebra and $\phi \in H^4(BG, \mathbb{Z})$ is a characteristic class corresponding to Φ under the Chern-Weil homomorphism. This differential character is uniquely defined when it satisfies the following:

- (1) The image of $S_{\Phi, \phi}(P, A)$ under the curvature map

$$\check{H}^3(M, U(1)) \rightarrow \Omega_{cl,0}^4(M, \mathbb{R})$$

is $\Phi\left(\frac{i}{2\pi}F_A\right)$ where F_A is the curvature form of A .

- (2) The image of $S_{\Phi, \phi}(P, A)$ under the characteristic class map $\check{H}^3(M, U(1)) \rightarrow H^4(M, \mathbb{Z})$ is $\phi(P)$, the characteristic class of P associated to ϕ .
- (3) The assignment of $S_{\Phi, \phi}(P, A)$ to (P, A) is natural with respect to morphisms of principal G -bundles with connection.

Since differential characters and bundle 2-gerbes with connection and curving on M are both classified by the Deligne cohomology group $H^3(M, \mathcal{D}^3)$, our Chern-Simons bundle 2-gerbe is a bundle gerbe version of the Cheeger-Simons invariant described above.

Given a connection \mathbb{A} on $\pi : EG \rightarrow BG$, the canonical Chern-Simons gauge theory c_ϕ associated to a class $\phi \in H^4(BG, \mathbb{Z})$ defines a universal Deligne characteristic class $c_\phi(EG, \mathbb{A})$ from the pair $(\phi, \Phi\left(\frac{i}{2\pi}F_{\mathbb{A}}\right))$. Then there is a Chern-Simons form $CS_\phi(\mathbb{A})$ associated to c_ϕ and (EG, \mathbb{A}) , satisfying

$$(6.1) \quad dCS_\phi(\mathbb{A}) = \pi^*\left(\Phi\left(\frac{i}{2\pi}F_{\mathbb{A}}\right)\right),$$

and the restriction of $CS_\phi(\mathbb{A})$ to a fiber of $EG \rightarrow BG$ determines a left-invariant closed 3-form ω_ϕ on G .

This universal Chern-Simons form $CS_\phi(\mathbb{A})$ can be constructed as in [24] via the pull-back principal G -bundle $\pi^*EG \rightarrow EG$, which admits a section $e \mapsto (e, e)$. This trivialisation defines a trivial connection \mathbb{A}_0 on π^*EG . Then $\pi^*\mathbb{A}$ and \mathbb{A}_0 defines a path of connections

$$\mathbb{A}_t = t\pi^*\mathbb{A} + (1-t)\mathbb{A}_0$$

for $0 \leq t \leq 1$, which can be thought as a connection on $[0, 1] \times \pi^*EG \rightarrow [0, 1] \times EG$. Define

$$(6.2) \quad CS_\phi(\mathbb{A}) = \int_{[0,1]} \Phi\left(\frac{i}{2\pi} F_{\mathbb{A}_t}\right).$$

Then the relation (6.1) follows from Stokes' theorem for the projection $[0, 1] \times EG \rightarrow EG$ and $\Phi\left(\frac{i}{2\pi} F_{\mathbb{A}_0}\right) = 0$ for \mathbb{A}_0 a trivial connection.

Remark 6.2. The corresponding left-invariant closed 3-form ω_ϕ on G is an integer multiple of the standard 3-form $\langle \theta, [\theta, \theta] \rangle$ where θ is the left-invariant Maurer-Cartan form on G and \langle , \rangle is the symmetric bilinear form on the Lie algebra of G defined by $\Phi \in I^2(G)$.

Given the fibration $EG \rightarrow BG$, we introduce the natural map $\hat{g} : EG^{[2]} \rightarrow G$ defined by $e_2 = e_1 \cdot \hat{g}(e_1, e_2)$ where $(e_1, e_2) \in EG^{[2]}$.

Definition 6.3. The *universal Chern-Simons bundle 2-gerbe* \mathcal{Q}_ϕ associated to $\phi \in H^4(BG, \mathbb{Z})$ is a bundle 2-gerbe $(\mathcal{Q}_\phi, EG^{[2]}; EG, BG)$ illustrated by the following diagram:

$$\begin{array}{ccc} & & \mathcal{G} \\ & \Downarrow & \downarrow \\ \mathcal{Q}_\phi & \xrightarrow{\hat{g}} & (G, \omega_\phi) \\ \Downarrow & & \\ EG^{[2]} & \xrightarrow[\pi_2]{\pi_1} & (EG, CS_\phi(\mathbb{A})) \\ & \downarrow \pi & \\ & & (BG, \Phi\left(\frac{i}{2\pi} F_{\mathbb{A}}\right)) \end{array}$$

where the bundle gerbe \mathcal{Q}_ϕ over $EG^{[2]}$ is obtained from the pull-back of a multiplicative bundle gerbe \mathcal{G} over G associated to $\tau(\phi) \in H^3(G, \mathbb{Z})$. Given a connection \mathbb{A} on EG , the bundle gerbe \mathcal{Q}_ϕ over $EG^{[2]}$ is equipped with a connection whose bundle gerbe curvature is given by $(\pi_2^* - \pi_1^*)CS_\phi(\mathbb{A})$, with $CS_\phi(\mathbb{A})$ the Chern-Simons form (6.2) on EG associated to ϕ and \mathbb{A} , and the bundle 2-gerbe curvature is given by $\Phi\left(\frac{i}{2\pi} F_{\mathbb{A}}\right)$. Moreover the bundle gerbe \mathcal{G} over G is equipped with a connection and curving with the bundle gerbe curvature given by ω_ϕ .

Proposition 6.4. *Given a class $\phi \in H^4(BG, \mathbb{Z})$, there exists a universal Chern-Simons bundle 2-gerbe over BG associated to a connection \mathbb{A} on EG .*

Proof. From the proof of Lemma 5.5, Proposition 5.7 and Theorem 5.8, we can represent a class $\phi \in H^4(BG, \mathbb{Z})$ by a bundle 2-gerbe $(\mathcal{Q}_\phi, X; EG, BG)$ over BG which is associated to the universal G -bundle $EG \rightarrow BG$ and to a multiplicative bundle gerbe \mathcal{G} over G , and is such that \mathcal{Q}_ϕ is stably isomorphic to $\hat{g}^*\mathcal{G}$ for $\hat{g} : EG^{[2]} \rightarrow G$. To complete the proof we have to equip $(\mathcal{Q}_\phi, X; EG, BG)$ with a bundle 2-gerbe connection and curving.

Let $g : P \rightarrow G$ be a gauge transformation on P . Denote by \mathbb{A}^g the connection on EG such that

$$\mathbb{A}^g(e) = \mathbb{A}(e \cdot g)$$

Then by direct calculation, we know that

$$CS_\phi(\mathbb{A}^g) = CS_\phi(\mathbb{A}) + g^*\omega_\phi - d\Phi(\mathbb{A}, dg \cdot g^{-1}),$$

where $CS_\phi(\mathbb{A})$ is the universal Chern-Simons form (6.2) associated to ϕ and (EG, \mathbb{A}) , and the left-invariant closed 3-form ω_ϕ on G (the transgression of $\Phi(\frac{i}{2\pi}F_{\mathbb{A}})$). Then under the map $\pi_2^* - \pi_1^*$, $CS_\phi(\mathbb{A})$ is mapped to a closed 3-form on $EG^{[2]}$ with periods in \mathbb{Z} . To see this note that $(\pi_2^* - \pi_1^*)CS_\phi(\mathbb{A})$ is closed, as

$$d(\pi_2^* - \pi_1^*)CS_\phi(\mathbb{A}) = (\pi_2^* - \pi_1^*) \circ \pi^*\Phi(\frac{i}{2\pi}F_{\mathbb{A}}) = 0,$$

and to confirm that $(\pi_2^* - \pi_1^*)CS_\phi(\mathbb{A})$ has its periods in \mathbb{Z} , we take a 3-cycle σ in $EG^{[2]}$, and form a 4-cycle in EG given by gluing two chains in EG with boundary $\pi_2(\sigma)$ and $-\pi_1(\sigma)$ as EG is contractible. Then $\Phi(\frac{i}{2\pi}F_{\mathbb{A}}) \in \Omega_{cl,0}^4(BG)$ implies that $(\pi_2^* - \pi_1^*)CS_\phi(\mathbb{A}) \in \Omega_{cl,0}^3(EG^{[2]})$. Actually, we have an explicit expression for $(\pi_2^* - \pi_1^*)CS_\phi(\mathbb{A})$, for $(e_1, e_2) \in EG^{[2]}$:

$$\begin{aligned} & (\pi_2^* - \pi_1^*)CS_\phi(\mathbb{A})(e_1, e_2) \\ &= CS_\phi(\mathbb{A})(e_2) - CS_\phi(\mathbb{A})(e_1) \\ &= CS_\phi(\mathbb{A})(e_1 \cdot \hat{g}(e_1, e_2)) - CS_\phi(\mathbb{A})(e_1) \\ &= \hat{g}^*\omega_\phi - d\Phi(\mathbb{A}, d\hat{g} \cdot \hat{g}^{-1}), \end{aligned}$$

from which we can see that ω_ϕ is a closed 3-form ω_ϕ on G with periods in \mathbb{Z} . Hence, we can choose a bundle gerbe connection and curving on the multiplicative bundle gerbe \mathcal{G} such that the bundle gerbe curvature is given by ω_ϕ . Moreover, we can choose a bundle gerbe connection and curving on $(\mathcal{Q}, X, EG^{[2]})$ whose bundle gerbe curvature is given by $(\pi_2^* - \pi_1^*)CS_\phi(\mathbb{A})$, hence the bundle 2-gerbe curving is given by the universal Chern-Simons form $CS_\phi(\mathbb{A})$.

On the Deligne cohomology level, we obtain a degree 2 Deligne class in $H^2(G, \mathcal{D}^2)$ associated to the multiplicative bundle gerbe \mathcal{G} with connection and curving over G and to the degree 3 Deligne class in $H^3(BG, \mathcal{D}^3)$ determined by $(\phi, \Phi(\frac{i}{2\pi}F_{\mathbb{A}}))$, the Deligne class for the bundle 2-gerbe \mathcal{Q} over BG with connection and curving. This completes the proof of the existence of a universal Chern-Simons bundle 2-gerbe associated to ϕ and (EG, \mathbb{A}) . \square

The upshot of all this is that the universal Chern-Simons bundle 2-gerbe over BG gives a geometric realization of the correspondence between three dimensional Chern-Simons gauge theories and Wess-Zumino-Witten models associated to G , from which the proof of Theorem 5.9 immediately follows.

Let $P \rightarrow M$ be a principal G -bundle with connection A . Let $f : M \rightarrow BG$ be a classifying map for this bundle with connection. This means that $f^*(EG, BG) \cong (P, M)$ and there exists a connection \mathbb{A} on EG such that $f^*\mathbb{A} = A$.

Definition 6.5. For the Chern-Simons gauge theory canonically defined by a class $\phi \in H^4(BG, \mathbb{Z})$, the *Chern-Simons bundle 2-gerbe* $\mathcal{Q}_\phi(P, A)$ associated with the principal G -bundle P with connection A over M is defined to be the pullback of the universal Chern-Simons bundle 2-gerbe by the classifying map f of (P, A) .

For a principal G -bundle $P \rightarrow M$ with a connection A , the corresponding Chern-Simons form for (P, A) corresponding to the class $\phi \in H^4(BG, \mathbb{Z})$ is

$$CS_\phi(A) = f^*CS_\phi(\mathbb{A}) \in \Omega^3(P),$$

such that

$$dCS_\phi(A) = \pi^*\Phi\left(\frac{i}{2\pi}F_A\right) \in \Omega_{cl,0}^4(M).$$

Hence, the curvature of the Chern-Simons bundle 2-gerbe $\mathcal{Q}_\phi(P, A)$ associated to (P, A) over M is given by $\Phi\left(\frac{i}{2\pi}F_A\right)$ and its bundle 2-gerbe curving is given by the Chern-Simons form $CS_\phi(A)$.

Remark 6.6. We can connect our approach to the familiar Chern-Simons action in the physics literature when $G = SU(N)$. We have a Deligne class constructed using the pullback construction from the universal Chern-Simons gauge theory

$$c_\phi(P, A) \in H^3(M, \mathcal{D}^3) \xrightarrow{(hol, curv)} \check{H}(M, U(1)).$$

Therefore for any smooth map σ from a closed 3-dimensional manifold Y to M , the holonomy of $c_\phi(P, A)$ associated to σ is given by

$$(6.3) \quad e^{2\pi i CS_\phi(\sigma; A)} = hol(c_\phi(P, A))(\sigma).$$

See also [24] for similar constructions. Now the Chern-Simons functional $CS_\phi(\sigma, A)$ will be the familiar formula when $\Phi = cw^{-1}(r(\phi))$ is the second Chern polynomial for $SU(N)$. Fix a trivialisation of (σ^*P, σ^*A) over Y , we can write the level k Chern-Simons functional as

$$CS(\sigma, A) = \frac{k}{8\pi^2} \int_Y Tr(\sigma^*A \wedge \sigma^*dA + \frac{1}{3}\sigma^*A \wedge \sigma^*A \wedge \sigma^*A).$$

Theorem 6.7. *With the canonical isomorphism between the Deligne cohomology and Cheeger-Simons cohomology, the Chern-Simons bundle 2-gerbe $\mathcal{Q}_\phi(P, A)$ is equivalent in Deligne cohomology to the Cheeger-Simons invariant $S_{\Phi, \phi}(P, A)$ described in Remark 6.1.*

Proof. It is well known that Cheeger-Simons differential characters are classified by Deligne cohomology (see, for example, [7]). Stable isomorphism classes of bundle 2-gerbes with connection and curving are also classified by Deligne cohomology (Proposition 4.8). The theorem of Cheeger and Simons given above defines a unique differential character satisfying certain conditions, thus it uniquely defines a class in Deligne cohomology and an equivalence class of bundle 2-gerbes with connection and curving, so we must show that the corresponding Deligne class associated to the Chern-Simons bundle 2-gerbe satisfies the required conditions.

- (1) The image under the map $H^3(M, \mathcal{D}^3) \rightarrow \Omega_0^4(M, \mathbb{R})$ is the curvature 4-form of the Chern-Simons bundle 2-gerbe. The curvature of the universal CS bundle 2-gerbe is given by $\Phi\left(\frac{i}{2\pi}F_A\right)$. Under pullback this becomes $\Phi\left(\frac{i}{2\pi}F_A\right)$.
- (2) The image of the map $H^3(M, \mathcal{D}^3) \rightarrow H^4(M, \mathbb{Z})$ is the characteristic 4-class of a bundle 2-gerbe. For the Chern-Simons bundle 2-gerbe, this shall be the pull-back of the 4-class of the universal Chern-Simons bundle 2-gerbe by the classifying map, which is by construction $\phi \in H^4(BG, \mathbb{Z})$.
- (3) If a principal G -bundle P_1 with connection A_1 , is related to another principal G -bundle with connection (P_2, A_2) , via a bundle morphism ψ then the corresponding classifying maps are related by $f_{P_1, A_1} = \psi \circ f_{P_2, A_2}$, so using both sides to pull back the universal CS bundle 2-gerbe we see that their corresponding Deligne classes behave as their Cheeger-Simons invariants $S_{\Phi, \phi}(P_1, A_1) = \psi^* S_{\Phi, \phi}(P_2, A_2)$.

□

Recall that for a connection on a line bundle over M , a gauge transformation is given by a smooth function $M \rightarrow U(1)$, and an extended gauge transformation for a bundle gerbe with connection and curving is given by a line bundle with connection over M . We can discuss extended gauge transformations for a bundle 2-gerbe with connection and bundle 2-gerbe curving over M .

The Chern-Simons bundle 2-gerbe $\mathcal{Q}_\phi(P, A)$ is equipped with a bundle 2-gerbe connection and curving such that the 2-curving is given by the Chern-Simons form $CS_\phi(A)$ with

$$dCS_\phi(A) = \pi^* \Phi\left(\frac{i}{2\pi}F_A\right) \in \Omega_{cl,0}^4(M).$$

Choose a covering $\{U_i\}$ of M , such that over U_i , $\pi : P \rightarrow M$ admits a section s_i , then we obtain a Čech representative of the Deligne class corresponding to $\mathcal{Q}_\phi(P, A)$

$$c_\phi(P, A) = [(g_{ijkl}, A_{ijk}, B_{ij}, C_i)]$$

with $C_i = s_i^*(CS_\phi(A))$. These local 3-forms $\{C_i\}$ are called the ‘C-field’ in string theory. We can say that our Chern-Simons bundle 2-gerbe $\mathcal{Q}_\phi(P, A)$ carries the Chern-Simons form as the ‘C-field’. As the bundle 2-gerbe curving (‘C-field’) is not uniquely determined, different choices are related by an extended gauge transformation. Suppose that $(g_{ijkl}, A_{ijk}, B_{ij}, C_i)$ represents the Deligne class of $\mathcal{Q}_\phi(P, A)$, then adding a term $(1, 0, 0, \omega)$ with a closed 3-form of integer period $\omega \in \Omega_{cl,0}^3(M)$ doesn’t change the Deligne class, following from the exact sequence (2.1). If M is 2-connected, we know that $\omega \in \Omega_{cl,0}^3(M)$ canonically defines a bundle gerbe with connection and curving over M whose bundle gerbe curvature is given by ω . We call this bundle gerbe an extended gauge transformation of the Chern-Simons bundle 2-gerbe.

Note that given another connection A' on $\pi : P \rightarrow M$, the Chern-Simons bundle 2-gerbe $\mathcal{Q}_\phi(P, A')$ is stably isomorphic to $\mathcal{Q}_\phi(P, A)$ as bundle 2-gerbes over M (they have the same characteristic class determined by ϕ). $\mathcal{Q}_\phi(P, A)$ and $\mathcal{Q}_\phi(P, A')$ have different bundle 2-gerbe curving, on the level of Deligne cohomology, the difference is given by

$$c_\phi(P, A) - c_\phi(P, A') = [(1, 0, 0, CS_\phi(A, A'))]$$

where $CS_\phi(A, A')$ is the well-defined Chern-Simons 3-form on M associated to a straight line path of connections on P connecting A and A' , (we remind that $CS_\phi(A)$ is well-defined only on P in general). It is natural (Cf.[18]) to define the so-called C -field on M to be a pair (A, c) where A is a connection on P and $c \in \Omega^3(M)$. So the space of C -fields is

$$\mathcal{A}_P \times \Omega^3(M)$$

where \mathcal{A}_P is the space of connections on $\pi : P \rightarrow M$. A C -field (A, c) canonically defines a degree 3 Deligne class

$$c_\phi(P, A) + [(1, 0, 0, c)] \in H^3(M, \mathcal{D}^3)$$

through the Deligne class $c_\phi(P, A)$ of the Chern-Simons bundle 2-gerbe $\mathcal{Q}_\phi(P, A)$. The gauge transformation group for the space of C -fields is defined to be

$$\Omega^1(M, adP) \times H^2(M, \mathcal{D}^2),$$

with the action on C -fields given by

$$(6.4) \quad (\alpha, \mathcal{D}) \cdot (A, c) = (A + \alpha, c + CS_\phi(A, A + \alpha) + curv(\mathcal{D}))$$

where $(\alpha, \mathcal{D}) \in \Omega^1(M, adP) \times H^2(M, \mathcal{D}^2)$, and $curv(\mathcal{D}) \in \Omega_{cl,0}^3(M)$ is the curvature of the degree 2 Deligne class \mathcal{D} (or the corresponding bundle gerbe with connection and curving) on M . Then it is easy to see that two C -fields that are gauge equivalent under (6.4) define the same degree 3 Deligne class on M through the Chern-Simons bundle 2-gerbe, hence, the same Cheeger-Simons differential character on M .

Remark 6.8. Denote by $\mathcal{G}(P)$ the gauge transformation group of P which acts on $\mathcal{A}_P \times \Omega^3(M)$ via $g \cdot (A, c) = (A^g, c + CS_\phi(A, A^g))$. Due to the fact that

$$CS_\phi(A, A^{g_1 g_2}) - CS_\phi(A, A^{g_2}) - CS_\phi(A^{g_2}, A^{g_1 g_2})$$

depends on the choice of A , this $\mathcal{G}(P)$ -action is not a group action. It is observed in [18] that if one interprets the space of C -fields with gauge group action (6.4) as an action groupoid, then $\mathcal{G}(P)$ -action is a sub-groupoid action.

7. MULTIPLICATIVE WESS-ZUMINO-WITTEN MODELS

In this section, we study the Wess-Zumino-Witten models in the image of the correspondence from $CS(G)$ to $WZW(G)$.

Let \mathcal{G} be a bundle gerbe with connection and curving over a compact Lie group G , whose Deligne class is in $H^2(G, \mathcal{D}^2)$. With the identification between the Deligne cohomology and the Cheeger-Simons cohomology (2.3),

$$(hol, curv) : H^2(G, \mathcal{D}^2) \rightarrow \check{H}^2(G, U(1))$$

where hol and $curv$ are the holonomy and the curvature maps for the Deligne cohomology, we will define the bundle gerbe holonomy for stable equivalence classes of bundle gerbes with connection and curving.

Let $\sigma : \Sigma \rightarrow G$ be a smooth map from a closed 2-dimensional surface Σ to G . σ represents a smooth 2-cocycle in $Z_2(G, \mathbb{Z})$. Define the holonomy of the bundle gerbe \mathcal{G} over G to be the holonomy of the corresponding Deligne class in $H^2(G, \mathcal{D}^2)$, denoted by $hol_{\mathcal{G}}$, then

$$hol_{\mathcal{G}}(\sigma) \in U(1),$$

is called the bundle gerbe holonomy $hol_{\mathcal{G}}(\cdot)$, evaluated on $\sigma : \Sigma \rightarrow G$.

We point out that as $H^2(G, \mathcal{D}^2)$ classifies stable isomorphism classes of bundle gerbes over G with connection and curving our bundle gerbe holonomy $hol_{\mathcal{G}}(\sigma)$ depends only on the stable isomorphism class of \mathcal{G} .

Proposition 7.1. *For a multiplicative bundle gerbe \mathcal{G} with connection and curving over G , the bundle gerbe holonomy satisfies the following multiplicative property:*

$$(7.1) \quad hol_{\mathcal{G}}(\sigma_1 \cdot \sigma_2) = hol_{\mathcal{G}}(\sigma_1) \cdot hol_{\mathcal{G}}(\sigma_2)$$

for any pair of smooth maps (σ_1, σ_2) from any closed surface Σ to G . Here $\sigma_1 \cdot \sigma_2$ denotes the smooth map from Σ to G obtained from the pointwise multiplication of σ_1 and σ_2 with respect to the group multiplication on G .

Proof. Recall our correspondence map in Definition 3.3 and our integration map (2.4). We constructed Ψ from a canonical G -bundle over $S^1 \times G$ in Definition 3.3 such that the bundle gerbe holonomy for the multiplicative bundle gerbe \mathcal{G} with connection and curving over G , being in the image of Ψ , corresponds to the bundle 2-gerbe holonomy for the Chern-Simons bundle 2-gerbe \mathcal{Q} over $S^1 \times G$ as follows. Given a smooth map $\sigma : \Sigma \rightarrow G$, $Id \times \sigma$ defines a smooth map $S^1 \times \Sigma \rightarrow S^1 \times G$, and

$$(7.2) \quad hol_{\mathcal{G}}(\sigma) = Hol_{\mathcal{Q}}(Id \times \sigma),$$

where $Hol_{\mathcal{Q}}$ denotes the bundle 2-gerbe holonomy for the Chern-Simons bundle 2-gerbe over $S^1 \times G$.

Given a pair of smooth maps σ_1 and σ_2 from any closed surface Σ to G . Denote by $\Sigma_{0,3}$ a fixed sphere with three holes. We can construct a flat G -bundle over $\Sigma_{0,3} \times \Sigma$ with boundary orientation given in such a way that the usual holonomies for flat G -bundle are σ_1 , σ_2 and $\sigma_1 \cdot \sigma_2$ respectively. The Chern-Simons bundle 2-gerbe associated to this flat G -bundle is a flat bundle 2-gerbe in the sense that the bundle 2-gerbe holonomy is a homotopy invariant (as follows from the exact sequence (2.2)). This implies that the bundle 2-gerbe holonomies for this flat Chern-Simons bundle 2-gerbe satisfy

$$(7.3) \quad Hol_{\mathcal{Q}}(Id \times \sigma_1 \cdot \sigma_2) = Hol_{\mathcal{Q}}(Id \times \sigma_1) \cdot Hol_{\mathcal{Q}}(Id \times \sigma_2).$$

Combining (7.2) and (7.3), we obtain the multiplicative property for the bundle gerbe holonomy of \mathcal{G} :

$$hol_{\mathcal{G}}(\sigma_1 \cdot \sigma_2) = hol_{\mathcal{G}}(\sigma_1) \cdot hol_{\mathcal{G}}(\sigma_2)$$

for any pair of smooth maps (σ_1, σ_2) from any closed surface Σ to G . \square

Recall that d_i ($i = 0, 1, 2$) are the face maps from $G \times G \rightarrow G$ such that $d_0(g_1, g_2) = g_2$, $d_1(g_1, g_2) = g_1 g_2$ and $d_2(g_1, g_2) = g_1$ for $(g_1, g_2) \in G$. Let \mathcal{G} be a bundle gerbe with connection and curving. Let $curv(\mathcal{G})$ be the bundle gerbe curvature of \mathcal{G} . We can consider the pair (σ_1, σ_2) as a map into $G \times G$. Hence we can define the holonomy of (σ_1, σ_2) with respect to the bundle gerbe connection and curving on

$$\delta(\mathcal{G}) = d_0^*(\mathcal{G}) \otimes d_1^*(\mathcal{G}^*) \otimes d_2^*(\mathcal{G}),$$

whose curvature is given by

$$d_0^*(curv(\mathcal{G})) - d_1^*(curv(\mathcal{G})) + d_2^*(curv(\mathcal{G})) = dB,$$

for a 2-form B on $G \times G$.

Because we have a trivialisation for $\delta(\mathcal{G})$, we can calculate this holonomy as

$$\exp \int_{\Sigma} (\sigma_1, \sigma_2)^* B.$$

On the other hand we can compose (σ_1, σ_2) with the three maps into G . This gives σ_1 , $\sigma_1 \cdot \sigma_2$ and σ_2 where the second of these maps is the result of pointwise multiplying in G . Because the bundle gerbe connection and curving on

$$\delta(\mathcal{G}) = d_0^*(\mathcal{G}) \otimes d_1^*(\mathcal{G}^*) \otimes d_2^*(\mathcal{G})$$

are δ of those on \mathcal{G} , then the bundle gerbe holonomy can also be calculated as

$$hol_{\mathcal{G}}(\sigma_1)hol_{\mathcal{G}}(\sigma_1\sigma_2)^{-1}hol_{\mathcal{G}}(\sigma_2)$$

so that we have

$$hol_{\mathcal{G}}(\sigma_1)hol_{\mathcal{G}}(\sigma_2) = \left(\exp \int_{\Sigma} (\sigma_1, \sigma_2)^* B \right) hol_{\mathcal{G}}(\sigma_1 \cdot \sigma_2).$$

The following proposition gives a necessary condition for a bundle gerbe with connection and curving to be multiplicative, and can be proved by direct calculation.

Proposition 7.2. *If a bundle gerbe \mathcal{G} with connection and curving on G is multiplicative, then $d_0^*\mathcal{G} \otimes d_2^*\mathcal{G}$ is stably isomorphic to $d_1^*\mathcal{G}$ as bundle gerbes over $G \times G$, and there exist a imaginary valued 2-form B on $G \times G$ such that*

$$\begin{aligned} d_0^*(curv(\mathcal{G})) - d_1^*(curv(\mathcal{G})) + d_2^*(curv(\mathcal{G})) &= dB, \\ \int_{\Sigma} (\sigma_1, \sigma_2)^* B &\in 2\pi i\mathbb{Z}, \end{aligned}$$

for any pair of smooth maps from any closed surface Σ to G .

For a Wess-Zumino-Witten model with group manifold G in the image of the correspondence map $\Psi : CS(G) \rightarrow WZW(G)$, we know that the bundle gerbe \mathcal{G} with connection and curving on G is multiplicative. The Wess-Zumino-Witten action regarded as a function on the space of smooth maps $\{\sigma : \Sigma \rightarrow G\}$ exponentiates to the bundle gerbe holonomy of \mathcal{G} , that is, for a smooth map σ , we have

$$\exp(S_{wzw}(\sigma)) = hol_{\mathcal{G}}(\sigma).$$

From Proposition 7.1, we know that the Wess-Zumino-Witten action for a multiplicative Wess-Zumino-Witten model satisfies the following property:

$$\exp(S_{wzw}(\sigma_1 \cdot \sigma_2)) = \exp(S_{wzw}(\sigma_1)) \cdot \exp(S_{wzw}(\sigma_2)),$$

for a pair of smooth maps σ_1 and σ_2 from any closed surface Σ to G .

From the commutative diagram:

$$\begin{array}{ccc} H^3(BG, \mathcal{D}^3) & \longrightarrow & H^2(G, \mathcal{D}^2) \\ \downarrow^c & & \downarrow^c \\ H^4(BG, \mathbb{Z}) & \rightarrow & H^3(G, \mathbb{Z}) \end{array}$$

we see that for a general compact semi-simple Lie group G , $H^3(BG, \mathcal{D}^3) \rightarrow H^2(G, \mathcal{D}^2)$ is not surjective. In particular, for non-simply connected compact semi-simple Lie group G , we know that the Wess-Zumino-Witten model on G is only multiplicative at certain levels. For example, the Wess-Zumino-Witten model on $SO(3)$ is multiplicative if and only if the Dixmier-Douady class of the corresponding bundle gerbe is an even class in $H^3(SO(3), \mathbb{Z}) \cong \mathbb{Z}$.

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